

ECE 6552 – Lecture 16

FEEDBACK LINEARIZATION 2

March 10 2025

Overview:

- Normal form for input-output feedback linearizable systems

Additional Reading:

- Khalil 13.2

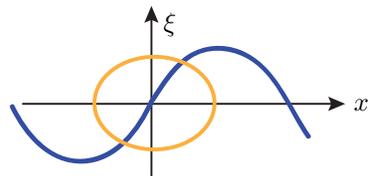
Feedback Linearization (continued)

Nonlinear Changes of Variables

$T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a *diffeomorphism* if its inverse T^{-1} exists, and both T and T^{-1} are continuously differentiable (C^1).

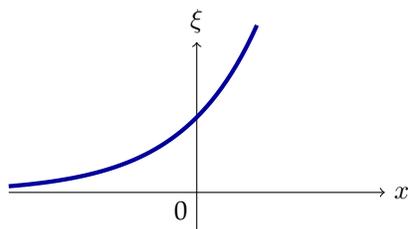
Examples:

1. $\xi = Tx$ is a diffeomorphism if T is a nonsingular matrix
2. $\xi = \sin x$ is a local diffeomorphism around $x = 0$, but not global

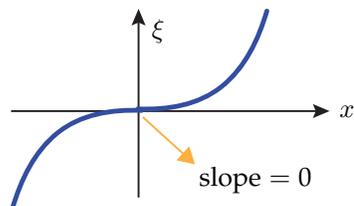


For a global inverse to exist, the function must be injective (one-to-one) and surjective (onto)

3. $\xi = e^x$ is a diffeomorphism from \mathbb{R} to $\mathbb{R}_{>0}$, but not from \mathbb{R} to \mathbb{R} .



4. $\xi = x^3$ is not a diffeomorphism because $T^{-1}(\cdot)$ is not C^1 at $\xi = 0$



How to check if $\xi = T(x)$ is a local diffeomorphism?

Implicit Function Theorem¹

Suppose $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is C^1 and there exists $x_0 \in \mathbb{R}^n, \xi_0 \in \mathbb{R}^n$ such that

$$f(x_0, \xi_0) = 0.$$

If $\frac{\partial f}{\partial x}(x_0, \xi_0)$ is nonsingular, then in a neighborhood of (x_0, ξ_0) ,

$$f(x, \xi) = 0$$

has a unique solution $x = g(\xi)$ where g is C^1 at $\xi = \xi_0$.

Corollary: Let $f(x, \xi) = T(x) - \xi$. If $\frac{\partial T}{\partial x}$ is nonsingular at x_0 , then $T(\cdot)$ is a local diffeomorphism around x_0 .

A "Normal Form" that Explicitly Displays the Zero Dynamics

Theorem: If $\dot{x} = f(x) + g(x)u, y = h(x)$ has a well-defined relative degree $r \leq n$, then there exist a diffeomorphism $T : x \mapsto \begin{bmatrix} z \\ \eta \end{bmatrix}$, $z \in \mathbb{R}^{n-r}, \eta \in \mathbb{R}^r$, that transforms the system to the form:

$$\begin{aligned} \dot{z} &= f_0(z, \eta) \\ \dot{\eta}_1 &= \eta_2 \\ &\vdots \\ \dot{\eta}_r &= \underbrace{b(z, \eta)}_{L_f^r(x)} + \underbrace{a(z, \eta)}_{L_g L_f^{r-1} h(x)} u, \quad y = \eta_1. \end{aligned} \tag{1}$$

In particular, $\dot{z} = f_0(z, 0)$ represents the zero dynamics. □

Finding the zero dynamics coordinates z :

To obtain this form, let $\eta = [h(x) \quad L_f h(x) \quad \dots \quad L_f^{r-1} h(x)]^T$. It remains to find $n - r$ independent variables z such that \dot{z} does not contain u .

One way to accomplish this is to find coordinates z_i (with $i = 1, \dots, n - r$) such that:

$$\frac{\partial z_i}{\partial x} g(x) = \left[\frac{\partial z_i}{\partial x_1} \dots \frac{\partial z_i}{\partial x_n} \right] g(x) \equiv 0$$

This can be interpreted as the coordinates z being chosen to be *normal* to the actuation vector $g(x)$.

¹ The key idea is that if a function has a regular Jacobian, then it locally behaves like a bijection with a smooth inverse

In summary, this change of coordinates is defined by the map:

$$T : x \mapsto \begin{bmatrix} z_1(x) \\ \vdots \\ z_{n-r}(x) \\ \eta_1(x) \\ \eta_2(x) \\ \vdots \\ \eta_r(x) \end{bmatrix} = \begin{bmatrix} z_1(x) \\ \vdots \\ z_{n-r}(x) \\ h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{bmatrix}, \quad DT = \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \cdots & \frac{\partial z_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_{n-r}}{\partial x_1} & \cdots & \frac{\partial z_{n-r}}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial L_f h}{\partial x_1} & \cdots & \frac{\partial L_f h}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial L_f^{r-1} h}{\partial x_1} & \cdots & \frac{\partial L_f^{r-1} h}{\partial x_n} \end{bmatrix} \quad (2)$$

- This map is a diffeomorphism if its Jacobian (DT) has full rank.
- Given a SISO system with a well-defined relative degree r . Then there exist coordinates $z_i, i = 1, \dots, n - r$ such that the map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diffeomorphism.

Example:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_3 + u \\ \dot{x}_3 &= \beta x_3 - u \\ y &= x_1. \end{aligned}$$

Let $\eta_1 = x_1, \eta_2 = x_2$. As we discovered last class, this results in the zero dynamics $\dot{x}_3 = (\alpha + \beta)x_3$.

To find the normal form, we need to find z such that \dot{z} does not contain u . This can be done by finding a solution to:

$$\frac{\partial z}{\partial x} g(x) = \begin{bmatrix} \frac{\partial z}{\partial x_1} & \frac{\partial z}{\partial x_2} & \frac{\partial z}{\partial x_3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0.$$

Thus, one choice is $z = x_2 + x_3$.

This z is independent of η_1, η_2 , and \dot{z} does not contain u . Plugging this in, the normal form is:

$$\dot{z} = (\alpha + \beta)x_3 = (\alpha + \beta)z - (\alpha + \beta) \underbrace{\eta_2}_{=0}$$

$$\dot{\eta}_1 = \eta_2$$

$$\dot{\eta}_2 = \alpha x_3 + u = \alpha z - \alpha \eta_2 + u.$$

I/O Linearizing Controller in the new coordinates (1):

$$u = \frac{1}{a(z, \eta)} \left(-b(z, \eta) + v \right) \quad (3)$$

$$v = -k_1 \eta_1 \cdots -k_r \eta_r \quad (4)$$

where k_1, \dots, k_r are such that all roots of $s^r + k_r s^{r-1} + \dots + k_2 s + k_1$ have negative real parts.

Theorem: If $z = 0$ is locally exponentially stable for the zero dynamics $\dot{z} = f_0(z, 0)$, then (3)–(4) locally exponentially stabilizes $x = 0$.

Proof: Closed-loop system:

$$\dot{z} = f_0(z, \eta)$$

$$\dot{\eta} = A\eta$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & \dots & \\ & & & \ddots & \\ & & & & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_r \end{bmatrix}$$

is Hurwitz. The Jacobian linearization at $(z, \eta) = 0$ is:

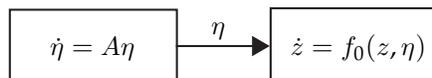
$$J = \begin{bmatrix} \frac{\partial f_0}{\partial z}(0, 0) & \frac{\partial f_0}{\partial \eta}(0, 0) \\ 0 & A \end{bmatrix}$$

where $\frac{\partial f_0}{\partial z}(0, 0)$ is Hurwitz since $\dot{z} = f_0(z, 0)$ is exponentially stable by the proposition in Lecture 12, page 2. Since A is also Hurwitz, all eigenvalues of J have negative real parts \Rightarrow exponential stability.

Global asymptotic stability can be guaranteed with additional assumptions on the zero dynamics, such as ISS of

$$\dot{z} = f_0(z, \eta)$$

with respect to the input ξ :



Example: $\dot{z} = -z + z^2 \eta, \quad \dot{\eta} = -k\eta$

Note: the z subsystem is not ISS

$(z, \eta) = 0$ is locally exponentially stable, but not globally: solutions escape in finite time for large $z(0)$.

I/O Linearizing Controller for Tracking

For the output $y(t)$ to track a reference signal² $y_d(t)$, replace (4) with:

² assumed to be r times differentiable

$$v = -k_1(\eta_1 - y_d(t)) - k_2(\eta_2 - \dot{y}_d(t)) \cdots - k_r(\eta_r - y_d^{(r-1)}(t)) + y_d^{(r)}(t)$$

Let $e_1 \triangleq \eta_1 - y_d(t)$, $e_2 \triangleq \eta_2 - \dot{y}_d(t)$, \dots , $e_r \triangleq \eta_r - y_d^{(r-1)}(t)$. Then:

$$\left. \begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ &\vdots \\ \dot{e}_r &= v - y_d^{(r)}(t) = -k_1 e_1 - \cdots - k_r e_r \end{aligned} \right\} \dot{e} = Ae.$$

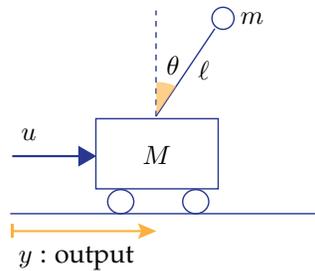
Thus $e(t) \rightarrow 0$, that is $y(t) - y_d(t) \rightarrow 0$.

If $y_d(t)$ and its derivatives are bounded, then $\eta(t)$ is bounded. If the zero dynamics $\dot{z} = f_0(z, \eta)$ is ISS with respect to η , then $z(t)$ is also bounded. Thus, all internal signals are bounded.

Cart-Pole Revisited

Example: Cart/Pole³

³ Simulation code for this example is available [online](#)



$$\begin{aligned} \ddot{y} &= \frac{1}{\frac{M}{m} + \sin^2 \theta} \left(\frac{u}{m} + \theta^2 \ell \sin \theta - g \sin \theta \cos \theta \right) \\ \ddot{\theta} &= \frac{1}{\ell \left(\frac{M}{m} + \sin^2 \theta \right)} \left(-\frac{u}{m} \cos \theta - \dot{\theta}^2 \ell \cos \theta \sin \theta + \frac{M+m}{m} g \sin \theta \right) \end{aligned} \quad (5)$$

In a bit more detail, if we select our system state to be $x = (p, \dot{p}, \theta, \dot{\theta})$, with p being the horizontal position of the cart, then our output is

$y = p$. The system dynamics can be explicitly written as:

$$\dot{x} = \begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{\frac{M}{m} + \sin^2 \theta} \left(\frac{u}{m} + \dot{\theta}^2 \ell \sin \theta - g \sin \theta \cos \theta \right) \\ x_4 \\ \frac{1}{\ell \left(\frac{M}{m} + \sin^2 \theta \right)} \left(-\frac{u}{m} \cos \theta - \dot{\theta}^2 \ell \cos \theta \sin \theta + \frac{M+m}{m} g \sin \theta \right) \end{bmatrix}$$

The relative degree of the system is 2, as we can see by differentiating the output twice with respect to time:

$$\begin{aligned} y &= p \\ \dot{y} &= \dot{p} \\ \ddot{y} &= \frac{1}{\frac{M}{m} + \sin^2 \theta} \left(\frac{u}{m} + \dot{\theta}^2 \ell \sin \theta - g \sin \theta \cos \theta \right) \end{aligned}$$

The input-output linearizing controller for this system would thus be:

$$u = -m \left(\dot{\theta}^2 \ell \sin \theta - g \sin \theta \cos \theta + v \left(\frac{M}{m} + \sin^2 \theta \right) \right)$$

where $v = -k_1(p - p_d) - k_2(\dot{p} - \dot{p}_d)$.

As we showed in last class, we can find the zero dynamics, by substituting $y = \dot{y} = 0$, and

$$u^* = -m(\dot{\theta}^2 \ell \sin \theta - g \sin \theta \cos \theta)$$

in the $\ddot{\theta}$ equation:

$$\ddot{\theta} = \frac{g}{\ell} \sin \theta.$$

Since the zero dynamics and output dynamics of our system are decoupled completely, the normal form can be written using the transformation:

$$T : x \mapsto \begin{bmatrix} z_1 \\ z_2 \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{bmatrix}$$