ECE 6552 – Lecture 12 [1]

TIME-VARYING SYSTEMS AND BACKSTEPPING

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Overview:

- Linear Time-Varying Systems
- Differential Lyapunov Equation
- Backstepping

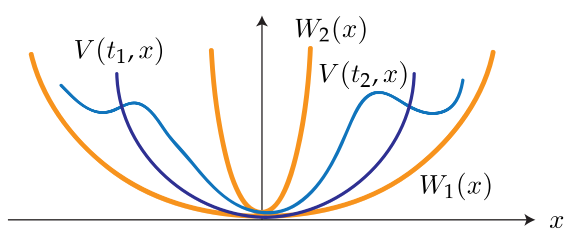
Additional Reading:

- Khalil, Chapter 8.3, 4.6
- Khalil, Chapter 14.3

Review of Lyapunov Stabilty Theorem for Time-Varying Systems

The Lyapunov stability theorems for time-varying systems introduced in the last lecture can be summarized as follows:

1. If $W_1(x) \le V(t,x) \le W_2(x)$ and $\dot{V}(t,x) \triangleq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le 0$ for some positive definite functions $W_1(\cdot)$, $W_2(\cdot)$ on a domain D that includes the origin, then $x = 0$ is uniformly stable.



2. If, further, $\dot{V}(t,x) \le -W_3(x)\ \forall x \in D$ for some positive definite $W_3(\cdot)$, then $x = 0$ is uniformly asymptotically stable.

3. If $D = \mathbb{R}^n$ and $W_1(\cdot)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.

4. If $W_i(x) = k_i|x|^a$, $i = 1, 2, 3$, for some constants $k_1, k_2, k_3, a > 0$, then $x = 0$ is uniformly exponentially stable.

Example:

$$\dot{x} = -g(t)x^3 \quad \text{where} \quad g(t) \ge 1 \quad \text{for all} \;\; t$$

$$V(x) = \frac{1}{2}x^2 \quad \Rightarrow \quad \dot{V}(t,x) = -g(t)x^4 \le -x^4 \triangleq W_3(x)$$

Globally uniformly asymptotically stable but not exponentially stable.

What if $W_3(\cdot)$ is only semidefinite?

Khalil, Section 8.3

Lasalle-Krasovskii Invariance Principle is not applicable to time-varying systems. Instead, we will have to use the following (weaker) result.

Theorem: . *Suppose* $W_1(x) \leq V(t,x) \leq W_2(x)$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x),$$

where $W_1(\cdot), W_2(\cdot)$ *are positive definite and* $W_3(\cdot)$ *is positive semidefinite. Suppose, further,* $W_1(\cdot)$ *is radially unbounded,* $f(t,x)$ *is locally Lipschitz in* x *and bounded in* t*, and* $W_3(\cdot)$ *is* C^1*. Then*

$$\boxed{W_3(x(t)) \to 0 \text{ as } t \to \infty.}$$

Note: This proves convergence to $S = \{x : W_3(x) = 0\}$ whereas the Invariance Principle, when applicable, guarantees convergence to the largest invariant set within S.

Linear Time-Varying Systems

Khalil Section 4.6, Sastry Section 5.7

Our linear time-varying system can be first introduced as simply a special case for our general time-varying system:

$$\dot{x} = A(t)x \qquad x(t) = \Phi(t,t_0)x(t_0) \tag{1}$$

The state transition matrix $\Phi(t,t_0)$ satisfies the equations:

$$\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0) \tag{2}$$

$$\frac{\partial}{\partial t_0}\Phi(t,t_0) = -\Phi(t,t_0)A(t_0) \tag{3}$$

- No eigenvalue test for stability in the time-varying case:
- For linear systems uniform asymptotic stability is equivalent to uniform exponential stability:

  Theorem: (4.11 in Khalil[2]). $x = 0$ *is uniformly asymptotically stable if and only if*

  $$\|\Phi(t,t_0)\| \leq ke^{-\lambda(t-t_0)} \text{ for some } k > 0,\ \lambda > 0.$$

  Example: $\dot{x} = A(t)x$. Take $V(x) = x^T P(t) x$:

  $$\begin{aligned}\dot{V}(x) &= x^T\dot{P}(t)x + \dot{x}^T P(t)x + x^T P(t)\dot{x} \\ &= x^T \underbrace{(\dot{P} + A^T P + PA)}_{\triangleq -Q(t)} x\end{aligned}$$

[2] Khalil Thm. 4.11, Sastry Thm. 5.33

If $k_1 I \leq P(t) \leq k_2 I$ and $k_3 I \leq Q(t)$, $k_1, k_2, k_3 > 0$, then

$$k_1|x|^2 \leq V(t,x) \leq k_2|x|^2 \quad \text{and} \quad \dot{V}(t,x) \leq -k_3|x|^2$$

$\Rightarrow$ global uniform exponential stability.

- $V(t,x) = x^T P(t) x$ proves uniform exp. stability if

$$\begin{array}{rl} \text{(i)} & \dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t) \\ \text{(ii)} & 0 < k_1 I \leq P(t) \leq k_2 I \\ \text{(iii)} & 0 < k_3 I \leq Q(t) \text{ for all } t. \end{array}$$

The converse is also true:

Theorem: Suppose $x = 0$ is uniformly exponentially stable, $A(t)$ is continuous and bounded, $Q(t)$ is continuous and symmetric, and there exist $k_3, k_4 > 0$ such that

$$0 < k_3 I \leq Q(t) \leq k_4 I \ \text{ for all } t.$$

Then, there exists a symmetric $P(t)$ satisfying (i)–(ii) above.

- For stable linear systems, there always exists quadratic Lyapunov functions
- Find them by choosing any positive definite $Q(t)$ and solve (differential) Lyapunov equation.

Proof:

Time-invariant: $P = \int_0^\infty e^{A^T\tau} Q e^{A\tau} d\tau$

Time-varying: $P(t) = \int_t^\infty \Phi^T(\tau,t) Q(\tau) \Phi(\tau,t) d\tau$

Using the Leibniz rule, property (3), and $\Phi(t,t) = I$ we obtain:

$$\begin{aligned} \dot{P}(t) &= \int_t^\infty \left(\frac{\partial}{\partial t}\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t) + \Phi^T(\tau,t)Q(\tau)\frac{\partial}{\partial t}\Phi(\tau,t) \right) d\tau \\ &\qquad - \Phi^T(t,t)Q(t)\Phi(t,t) \\ &= \int_t^\infty \left(-A^T(t)\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t) - \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)A(t) \right) d\tau \\ &\qquad - \Phi^T(t,t)Q(t)\Phi(t,t) \\ &= -A^T(t)P(t) - P(t)A(t) - Q(t). \end{aligned}$$

Lyapunov-based Feedback Design

We have now covered all components of Lyapunov theory. We will transition to introducing how Lyapunov theory can be used to design control laws. In particular we will introduce backstepping (today's lecture)and Control Lyapunov Functions (in future lectures). Another example that we will not cover in class is adaptive control (MRAC to be exact).

Backstepping

Khalil (Sec. 14.3), Sastry (Sec. 6.8)

Feedback stabilization: Given a control-affine nonlinear system[3]

$$\dot{x} = f(x) + g(x)u \tag{4}$$

[3] We will show in a later lecture that all mechanical (robotic) systems can be cast in this form.

with input $u \in \mathbb{R}$ and smooth functions $f : D \to \mathbb{R}^n$ and $g : D \to \mathbb{R}^n$, design a control law $u = k(x)$ such that $x = 0$ is asymptotically stable for the closed-loop system:

$$\dot{x} = f(x) + g(x)k(x).$$

Backstepping is a technique that simplifies this task for a class of systems.

Suppose a stabilizing feedback $u = k(\eta)$, $k(0) = 0$, is available for:

$$\dot{\eta} = F(\eta) + G(\eta)u \quad \eta \in \mathbb{R}^n, u \in \mathbb{R},\ F(0) = 0,$$

along with a Lyapunov function V such that

$$\frac{\partial V}{\partial \eta}\Big(F(\eta) + G(\eta)k(\eta)\Big) \leq -W(\eta) < 0 \quad \forall \eta \neq 0.$$

Can we modify $k(\eta)$ to stabilize the augmented system below?

$$\boxed{\begin{aligned}\dot{\eta} &= F(\eta) + G(\eta)\xi \\ \dot{\xi} &= u.\end{aligned}}$$

Define the error variable $\boxed{z = \xi - k(\eta)}$ and change variables: $(\eta, \xi) \to (\eta, z)$:

$$\begin{aligned}\dot{\eta} &= F(\eta) + G(\eta)k(\eta) + G(\eta)z \\ \dot{z} &= u - \dot{k}(\eta, z)\end{aligned}$$

where $\dot{k}(\eta, z) = \frac{\partial k}{\partial \eta}\Big(F(\eta) + G(\eta)k(\eta) + G(\eta)z\Big)$. Take the new Lyapunov function:

$$V_+(\eta, z) = V(\eta) + \frac{1}{2}z^2.$$

$$\begin{aligned}\dot{V}_+ &= \frac{\partial V}{\partial \eta}\dot{\eta} + z\dot{z} \\ &= \underbrace{\frac{\partial V}{\partial \eta}\Big(F(\eta) + G(\eta)k(\eta)\Big)}_{\leq -W(\eta)} + \underbrace{\frac{\partial V}{\partial \eta}G(\eta)z + z(u - \dot{k})}_{= z\left(u - \dot{k} + \frac{\partial V}{\partial \eta}G(\eta)\right)}\end{aligned}$$

Let: $\boxed{u = \dot{k} - \frac{\partial V}{\partial \eta}G(\eta) - Kz, \quad K > 0.}$

Then, $\dot{V}_+ \leq -W(\eta) - Kz^2 \;\Rightarrow\; (\eta, z) = 0$ is asymptotically stable.

- Above we discussed backstepping over a pure integrator. The main idea generalizes trivially to:

$$\boxed{\begin{aligned}\dot{\eta} &= F(\eta) + G(\eta)x \\ \dot{x} &= f(\eta, x) + g(\eta, x)u\end{aligned}}$$

where $\eta \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $g(\eta, x) \neq 0$ for all $(\eta, x) \in \mathbb{R}^{n+1}$.

With the preliminary feedback

$$u = \frac{1}{g(\eta, x)}(-f(\eta, x) + v) \tag{5}$$

the x-subsystem becomes a pure integrator: $\dot{x} = v$. Substituting the backstepping control law from above:

$$v = \dot{k} - \frac{\partial V}{\partial \eta}G(\eta) - Kz, \quad z \triangleq x - k(\eta), \quad K > 0$$

into (5), we get:

$$\boxed{u = \frac{1}{g(\eta, x)}\left(-f(\eta, x) + \dot{k} - \frac{\partial V}{\partial \eta}G(\eta) - Kz\right).}$$

- Backstepping can be applied recursively to systems of the form:[4]

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\ &\vdots \\ \dot{x}_n &= f_n(x) + g_n(x)u\end{aligned} \tag{6}$$

where $g_i(x_1, \ldots, x_i) \neq 0$ for all $x \in \mathbb{R}^n$, $i = 2, 3, \cdots, n$.

[4] Systems of this form are called "strict feedback systems."

Example:

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = u$$

Treat x_2 as "virtual" control input for the x_1-subsystem:

$$k(x_1) = -Kx_1 - x_1^2 \quad K > 0$$
$$V_1(x_1) = \frac{1}{2}x_1^2.$$

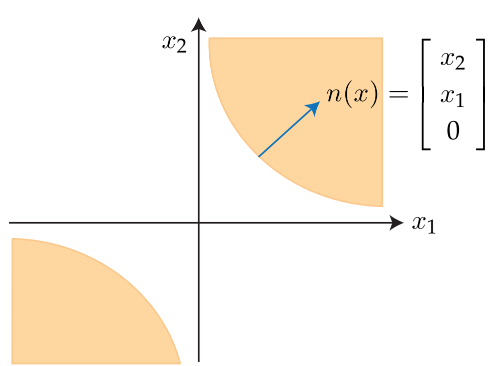
Apply backstepping:

$$z_2 = x_2 - k(x_1) = x_2 + Kx_1 + x_1^2$$
$$\dot{z}_2 = u - \dot{k}$$
$$u = \dot{k} - \frac{\partial V_1}{\partial x_1} - k_2 z_2, \quad k_2 > 0$$
$$= \underbrace{-(K + 2x_1)(x_1^2 + x_2)}_{=\dot{k}} - \underbrace{x_1}_{=\frac{\partial V_1}{\partial x_1}} - k_2\underbrace{(x_2 + Kx_1 + x_1^2)}_{=z_2}.$$

Example 2:

$$\begin{aligned} \dot{x}_1 &= (x_1x_2 - 1)x_1^3 + (x_1x_2 + x_3^2 - 1)x_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u. \end{aligned} \tag{7}$$

Not in strict feedback form because x_3 appears too soon. In fact, this system is not globally stabilizable because the set $x_1x_2 \geq 2$ is positively invariant regardless of u:



To see this, note that

$$n(x) \cdot f(x,u) = [(x_1 x_2 - 1)x_1^3 + (x_1 x_2 + x_3^2 - 1)x_1]x_2 + x_3 x_1$$

and substitute $x_1 x_2 = 2$:

$$\begin{aligned} &= \left(x_1^3 + (1 + x_3^2)x_1\right)x_2 + x_3 x_1 \\ &= \left(x_1^2 + (1 + x_3^2)\right)x_1 x_2 + x_3 x_1 \\ &= 2x_1^2 + 2(1 + x_3^2) + x_3 x_1 \\ &= \underbrace{2x_1^2 + x_3 x_1 + 2x_3^2}_{\geq 0} + 2 > 0. \end{aligned}$$

Example 3:

$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{8}$$

Treat x_2 as virtual control and let $\boxed{\alpha_1(x_1) = -x_1}$ which stabilizes the x_1-subsystem, as verified with Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$.
Then $z_2 := x_2 - \alpha_1(x_1)$ satisfies $\dot{z}_2 = u - \dot{\alpha}_1$, and

$$u = \dot{\alpha}_1 - \frac{\partial V_1}{\partial x_1} x_1^2 - k_2 z_2 = -x_1^2 x_2 - x_1^3 - k_2(x_2 + x_1)$$

achieves global asymptotic stability:

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \quad \Rightarrow \quad \dot{V} = -x_1{}^4 - k_2 z_2^2.$$

Note that we can't conclude exponential stability due to the quartic term x_1^4 above (recall the Lyapunov sufficient condition for exponential stability in Lecture 11, p.2). In fact, the linearization of the closed-loop system proves the lack of exponential stability:

$$\begin{bmatrix} 0 & 0 \\ 0 & -k_2 \end{bmatrix} \rightarrow \lambda_{1,2} = 0, -k_2.$$