ECE 6552 – Lecture 11 [1]

TIME-VARYING SYSTEMS

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Overview:

- Introduce time-varying systems and comparison functions
- Alternative statement of Lyapunov
- Lyapunov theory in time-varying systems

Additional Reading:

- Khalil, Chapter 4.4-4.6

Time-Varying Systems

Khalil (Sec. 4.5), Sastry (Sec. 5.2)

$$\dot{x} = f(t, x) \quad f(t, 0) \equiv 0 \tag{1}$$

The following definitions of stability and asymptotic stability will require a class of functions called "comparison functions". We will begin with the definitions of stability and later introduce these functions.

Time-Varying Stability Definitions

Definition: $x = 0$ is stable if for every $\varepsilon > 0$ and t_0, there exists $\delta > 0$ such that

$$|x(t_0)| \leq \delta(t_0, \varepsilon) \implies |x(t)| \leq \varepsilon \quad \forall t \geq t_0.$$

If the same δ works for all t_0, *i.e.* $\delta = \delta(\varepsilon)$, then $x = 0$ is uniformly stable.

It is easier to define uniform stability and uniform asymptotic stability using comparison functions:

- $x = 0$ is uniformly stable if there exists a class-$\mathcal{K}$ function $\alpha(\cdot)$ and a constant $c > 0$ such that

  $$|x(t)| \leq \alpha(|x(t_0)|)$$

  for all $t \geq t_0$ and for every initial condition such that $|x(t_0)| \leq c$.

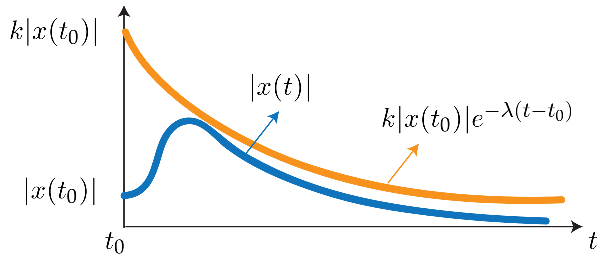
- uniformly asymptotically stable if there exists a class-$\mathcal{KL}$ $\beta(\cdot, \cdot)$ s.t.

  $$|x(t)| \leq \beta(|x(t_0)|, t - t_0)$$

  for all $t \geq t_0$ and for every initial condition such that $|x(t_0)| \leq c$.

- globally uniformly asymptotically stable if $c = \infty$.

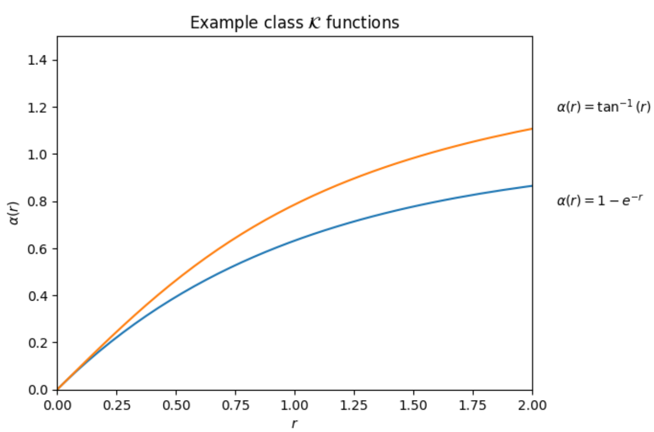
- uniformly exponentially stable if $\beta(r,s) = kre^{-\lambda s}$ for some $k, \lambda > 0$:
$$|x(t)| \leq k|x(t_0)|e^{-\lambda(t-t_0)}$$
for all $t \geq t_0$ and for every initial condition such that $|x(t_0)| \leq c$.
$k > 1$ allows for overshoot:



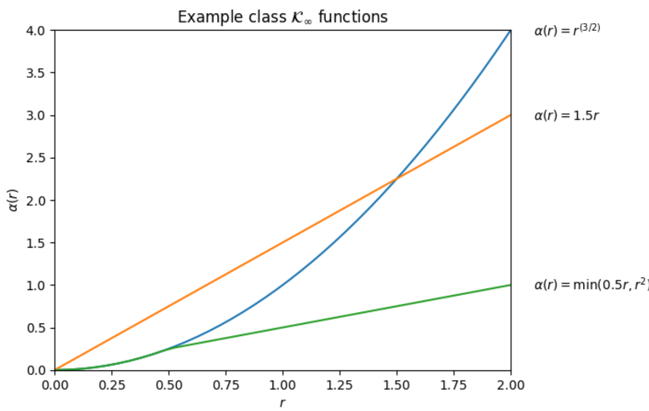
Comparison Functions

Example function code to produce the example plots is provided online.

Definition: *Class-$\mathcal{K}$.* A continuous function $\alpha : [0,a) \to [0,\infty)$ is class-$\mathcal{K}$ (denoted $\alpha \in \mathcal{K}$) if $\alpha(0) = 0$ and strictly monotonic (i.e., zero at zero and strictly increasing).

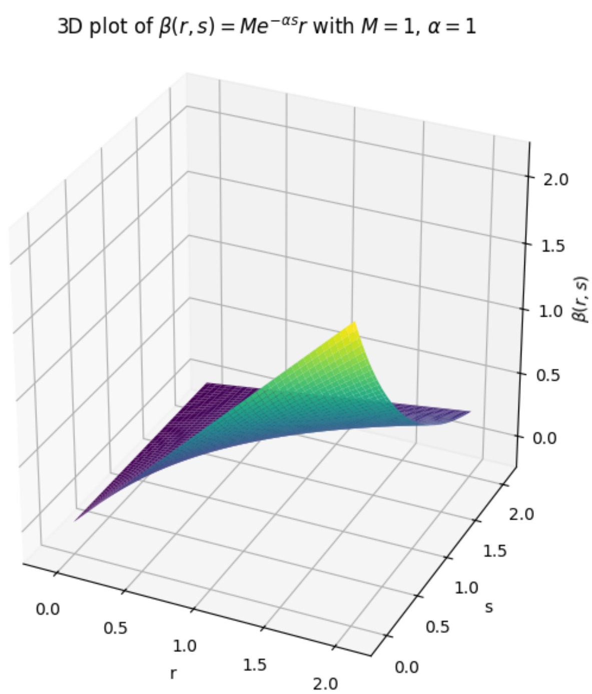


Definition: *Class-$\mathcal{K}_\infty$.* A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is class-$\mathcal{K}_\infty$ (denoted $\alpha \in \mathcal{K}_\infty$) if it is class-$\mathcal{K}$ and $\alpha(r) \to \infty$ as $r \to \infty$.



Definition: *Class-$\mathcal{KL}$.* A continuous function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is class-$\mathcal{KL}$ (denoted $\beta \in \mathcal{KL}$) if:

1. $\beta(\cdot, s)$ is class-$\mathcal{K}$ for every fixed s.
2. $\beta(r, \cdot)$ is decreasing and $\beta(r, s) \to 0$ as $s \to \infty$ for every fixed r.



Example: $\alpha(r) = \tan^{-1}(r)$ is class-$\mathcal{K}$, $\alpha(r) = r^c, c > 0$ is class-$\mathcal{K}_\infty$, $\beta(r, s) = r^c e^{-s}$ is class-$\mathcal{KL}$.

Proposition: If $V(\cdot)$ is positive definite, then we can find class-$\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|). \tag{2}$$

If $V(\cdot)$ is radially unbounded, we can choose $\alpha_1(\cdot)$ to be class-$\mathcal{K}_\infty$.

Example: $V(x) = x^T P x \quad P = P^T > 0$

$$\alpha_1(|x|) = \lambda_{\min}(P)|x|^2 \quad \alpha_2(|x|) = \lambda_{\max}(P)|x|^2.$$

Examples of Time-Varying Stability

Example: Consider the following system, defined for $t > -1$:

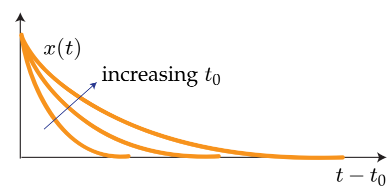
$$\dot{x} = \frac{-x}{1+t} \tag{3}$$

$$\begin{aligned} x(t) &= x(t_0)e^{\int_{t_0}^{t} \frac{-1}{1+s}ds} = x(t_0)e^{\log(1+s)|_t^{t_0}} \\ &= x(t_0)e^{\log\frac{1+t_0}{1+t}} = x(t_0)\frac{1+t_0}{1+t} \end{aligned}$$

$|x(t)| \le |x(t_0)| \implies$ the origin is uniformly stable with $\alpha(r) = r$.

The origin is also asymptotically stable, but not uniformly, because the convergence rate depends on t_0:

$$x(t) = x(t_0)\frac{1+t_0}{1+t_0+(t-t_0)} = \frac{x(t_0)}{1+\frac{t-t_0}{1+t_0}}.$$



Example:

$$\dot{x} = -x^3 \quad \Rightarrow \quad x(t) = \text{sgn}(x(t_0))\sqrt{\frac{x_0^2}{1+2(t-t_0)x_0^2}}$$

$x = 0$ is asymptotically stable but not exponentially stable.

Proposition: $x = 0$ is exponentially stable for $\dot{x} = f(x)$, $f(0) = 0$, if and only if $A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=0}$ is Hurwitz, that is $\Re\lambda_i(A) < 0\ \forall i$.

Although strict inequality in $\Re\lambda_i(A) < 0$ is not necessary for asymptotic stability, it *is* necessary for exponential stability.

Modern Statement of Lyapunov

One benefit of class $\mathcal{K}$ functions is that they allow us to state Lyapunov's theorem in a more modern form. In essence, this results in a 1-dimensional dynamical system:

$$\dot{V} \leq -\alpha(V)$$

with $\alpha \in \mathcal{K}$ which can be shown (it relies on the Comparison Lemma which is below) to imply that V evolves according to a class $\mathcal{KL}$ function (i.e., $V(t) \leq \beta(V(0), t)$).

Theorem: Modern Statement of Lyapunov's Theorem. *Let $\dot{x} = f(x)$ where $f : D \to \mathbb{R}^n$ is C^1 and D is a neighborhood of the origin with $f(0) = 0$. Consider the function $V : D \to \mathbb{R}$ that is C^1 and satisfies $V(0) = 0$. If the following conditions are satisfied:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$
$$\dot{V} \leq -\alpha_3(\|x\|)$$

for $\alpha_i \in \mathcal{K}$, $i = 1, 2, 3$, then $x = 0$ is asymptotically stable. Moreover,

$$\|x(t)\| \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(0)\|), t - t_0)), \quad \forall t \geq t_0$$

where $\beta \in \mathcal{KL}$ is the solution of the IVP

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \quad y(t_0) = V(x(t_0)).$$

Note: The proof for the modern statement of Lyapunov's Theorem follows from the Comparison Lemma. Details can be found in Khalil, Section 4.4.

Lemma: Comparison Lemma. *Let $\alpha \in \mathcal{K}$ be a class-$\mathcal{K}$ function on some interval with $a \in \mathbb{R}_{>0}$. Consider the IVP:*

$$\dot{u} = -\alpha(u), \quad u(0) = u_0,$$

and assume that it has a unique solution[2] $u(t)$ on $[0, a]$. If $v : [0, a] \to \mathbb{R}$ is C^1 satisfying:

$$\dot{v} \leq -\alpha(v(t)), \quad v(0) \leq u_0,$$

then $v(t) \leq u(t)$ for all $t \in [0, a]$.

[2] Recall that $u(t)$ will have a unique solution if $f(\cdot)$ is Lipschitz continuous. Thus sometimes the condition for the modern Lyapunov Theorem is stated as f is locally Lipschitz continuous.

The proof of this comparison lemma follows from the following composition rules for class $\mathcal{K}$ functions (Lemma 4.2 in Khalil):

Lemma: (4.2 in Khalil). *Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a)$, α_3 and α_4 be class $\mathcal{K}_\infty$, and β be a class $\mathcal{KL}$ function. We will denote the inverse of α_i by α_i^{-1}. Then the following hold:*

1. *α_1^{-1} is defined on $[0, \alpha_1(a))$ and belongs to class $\mathcal{K}$.*

2. α_3^{-1} *is defined on* $[0, \infty)$ *and belongs to class* $\mathcal{K}_\infty$.
3. $\alpha_1 \circ \alpha_2$ *belongs to class* $\mathcal{K}$.
4. $\alpha_3 \circ \alpha_4$ *belongs to class* $\mathcal{K}_\infty$.
5. $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ *belongs to class* $\mathcal{KL}$.

Lyapunov's Stability Theorem for Time-Varying Systems

Khalil, Section 4.5

Theorem: (4.8 in Khalil). *Let $x = 0$ be an equilibrium point for the system $\dot{x} = f(t, x)$, where $f : [0, \infty) \times D \to \mathbb{R}^n$ is locally Lipschitz in x on $[0, \infty) \times D$ and $D \subset \mathbb{R}^n$ contains the origin. Let $V : [0, \infty) \times D \to \mathbb{R}$ be C^1 such that:*

$$W_1(x) \leq V(t, x) \leq W_2(x)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

$\forall t \geq 0$ and $\forall x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then $x = 0$ is uniformly stable.

A continuous positive definite function is a relaxation of a class-$\mathcal{K}$ function

This theorem can be extended to show uniform asymptotic stability:

Theorem: (4.9 in Khalil). *Suppose the assupmtions of Theorem 4.8 are satisfied, with the inequality strengthened to*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$$

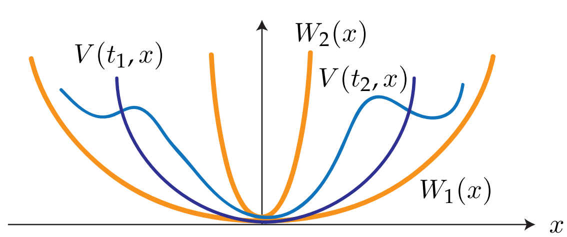
$\forall t \geq 0$ and $\forall x \in D$ where $W_3(x)$ is a continuous positive definite function on D. Then, $x = 0$ is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \leq r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r \mid W_2(x) \leq c\}$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0$$

for some class $\mathcal{KL}$ function β. Finally, if $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded, then $x = 0$ is globally uniformly asyumptotically stable.

These theorems can be summarized as follows:

1. If $W_1(x) \leq V(t, x) \leq W_2(x)$ and $\dot{V}(t, x) \triangleq \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$ for some positive definite functions $W_1(\cdot)$, $W_2(\cdot)$ on a domain D that includes the origin, then $x = 0$ is uniformly stable.



2. If, further, $\dot{V}(t,x) \leq -W_3(x)\ \forall x \in D$ for some positive definite $W_3(\cdot)$, then $x = 0$ is uniformly asymptotically stable.

3. If $D = \mathbb{R}^n$ and $W_1(\cdot)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.

4. If $W_i(x) = k_i|x|^a$, $i = 1, 2, 3$, for some constants $k_1, k_2, k_3, a > 0$, then $x = 0$ is uniformly exponentially stable.

Example:

$$\dot{x} = -g(t)x^3 \quad \text{where} \quad g(t) \geq 1 \quad \text{for all} \;\; t$$

$$V(x) = \frac{1}{2}x^2 \quad \Rightarrow \quad \dot{V}(t,x) = -g(t)x^4 \leq -x^4 \triangleq W_3(x)$$

Globally uniformly asymptotically stable but not exponentially stable. Take $g(t) \equiv 1$ as a special case:

$$\dot{x} = -x^3 \quad \Rightarrow \quad x(t) = \text{sgn}(x(t_0))\sqrt{\frac{x_0^2}{1 + 2(t - t_0)x_0^2}}$$

which converges slower than exponentially.

What if $W_3(\cdot)$ is only semidefinite?

Khalil, Section 8.3

Lasalle-Krasovskii Invariance Principle is not applicable to time-varying systems. Instead, we must use a (weaker) result. This will be discussed in the next Lecture.