

## ME 6402 – Lecture 25

### HIGHER-ORDER CONTROL BARRIER FUNCTIONS

April 10 2025

Overview:

- Introduce the notion of *relative degree* for control barrier functions
- Extend CBFs to systems with relative degree  $> 1$

Additional Reading:

- A. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, “Control Barrier Functions: Theory and Applications,” IEEE Transactions on Automatic Control, 2019.

### Control Barrier Functions

**Definition: CBF (recall).** A function  $h$  with  $\mathcal{C} = \{x \mid h(x) \geq 0\}$  is a *control barrier function* (CBF) for  $\dot{x} = f(x) + g(x)u$  if there exists a locally Lipschitz function  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $\alpha(0) = 0$  such that

$$\sup_{u \in \mathbb{R}^m} \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x)) \quad \text{for all } x \in \mathbb{R}^n. \quad (1)$$

We can also write (1) using Lie derivative notation:

$$\sup_{u \in \mathbb{R}^m} L_f h(x) + L_g h(x)u \geq -\alpha(h(x)) \quad (2)$$

Define

$$U(x) = \{u \in \mathbb{R}^m \mid \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))\}. \quad (3)$$

**Theorem: CBF (recall).** If  $h$  is a control barrier function for  $\dot{x} = f(x) + g(x)u$ , then the following hold:

1.  $U(x) \neq \emptyset$  for all  $x$ ;
2. Any Lipschitz feedback control  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  satisfying  $u(x) \in U(x)$  renders  $\mathcal{C}$  invariant;
3. A feedback control is given by

$$u^*(x) = \begin{cases} 0 & \text{if } L_f h(x) + \alpha(h(x)) \geq 0 \\ \frac{-(L_f h(x) + \alpha(h(x)))L_g h(x)^T}{L_g h(x)L_g h(x)^T} & \text{otherwise.} \end{cases} \quad (4)$$

For  $u^*(x)$  to be Lipschitz on some domain, we must certify that  $L_g h(x) \neq 0$  everywhere on the domain.

**Example 1:** (Cart-Pole System Revisited)

Recall the model of the cart-pole system:

$$\begin{aligned}\ddot{p} &= \frac{1}{1 + \sin^2 \theta} \left( u + \dot{\theta}^2 \sin \theta - g \sin \theta \cos \theta \right) \\ \ddot{\theta} &= \frac{1}{1 + \sin^2 \theta} \left( -u \cos \theta - \dot{\theta}^2 \cos \theta \sin \theta + 2g \sin \theta \right)\end{aligned}\tag{5}$$

Unlike last lecture, suppose we want  $y$  to satisfy  $-L \leq p \leq L$ . Try

$$h(x) = \frac{1}{2}(-p^2 + L^2)\tag{6}$$

$$\alpha(s) = \gamma s, \quad \gamma > 0.\tag{7}$$

But  $\nabla h(x)^T g(x) \equiv 0$ . Then  $h$  cannot be a CBF because the control input vanishes from the CBF condition:

$$\sup_{u \in \mathbb{R}^m} L_f h(x) + L_g h(x)u \geq -\alpha(h(x))$$

For systems such that  $\dot{h}(x)$  does *not* depend on  $u$ , we need  $h$  that depends on more state variables. There is a systematic way to do this. Suppose  $h$  satisfies  $L_g h(x) \equiv 0$  and cannot be used as a CBF. Define:

$$\Psi_1(x) = L_f h(x) + \alpha_1(h(x))$$

for some Lipschitz  $\alpha_1$  satisfying  $\alpha_1(0) = 0$ , and let

$$\mathcal{C}_1 = \{x \mid \Psi_1(x) \geq 0\}$$

This higher-order CBF is then enforced by the condition:

$$\sup_{u \in \mathbb{R}^m} L_f \Psi_1(x) + L_g \Psi_1(x)u \geq -\alpha_2(\Psi_1(x))$$

**Lemma: Higher-Order CBF Invariance.** Suppose  $u(x)$  is a feedback control law such that  $\mathcal{C}_1$  is invariant. Then  $\mathcal{C} \cap \mathcal{C}_1$  is also invariant, where  $\mathcal{C} = \{x \mid h(x) \geq 0\}$ .

*Proof.* Consider  $x_0 \in \mathcal{C} \cap \mathcal{C}_1$  and let  $x(t)$  be a corresponding closed-loop trajectory. Then  $x(t) \in \mathcal{C}_1$  for all  $t \geq 0$  by assumption, and therefore

$$\dot{h}(x(t)) = L_f h(x(t)) \geq -\alpha_1(h(x(t))).$$

Since  $h(x_0) \geq 0$  by assumption,  $h(x(t)) \geq 0$  for all  $t \geq 0$  by the Comparison Lemma.  $\square$

Question: How can we ensure that  $\mathcal{C}_1$  is invariant?

Answer: Use  $\Psi_1(x)$  as a CBF!

- If  $\nabla \Psi_1(x)^T g(x) \equiv 0$ , repeat the process, defining  $\Psi_2(x) = \nabla \Psi_1(x)^T f(x) + \alpha_2(\Psi_1(x))$ .
- $h(x)$  is called a *high-order CBF of degree  $r$*  when this process ends with a CBF  $\Psi_{r-1}(x)$ .

How many times will we repeat, i.e., what is  $r$ ? This is related to relative degree.

- *Least relative degree  $r$*  is the minimum relative degree over all states  $x$ . Therefore  $L_g \Psi_{r-1}(x) = L_g L_f^{r-2} h(x) \neq 0$  for some  $x$ , but not necessarily all  $x$ .

For the previous construction to lead to a valid CBF, we need:

$$L_f \Psi_{r-1}(x) + \alpha_r(\Psi_r(x)) \geq 0, \quad \text{whenever } L_g \Psi_{r-1}(x) = 0$$

- States where  $L_g \Psi_{r-1}(x) = 0$  become important to pay attention to (more on this later)

**Example 2:** Consider the double integrator  $\dot{x}_1 = u$ , i.e.,  $\dot{x}_1 = x_2$ ,  $\dot{x}_2 = u$ . Explicitly, this is written in control-affine form:

$$\dot{x} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Suppose that we want  $x_1 \leq L$  always. Choose:

$$h(x) = L - x_1$$

We can check the relative degree of the control barrier function:

$$\nabla h(x)^T g(x) = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv 0$$

This is the same thing as differentiating  $h(x)$  until  $u$  appears:

$$\begin{aligned} \dot{h}(x) &= -\dot{x}_1 = -x_2 \\ \ddot{h}(x) &= -\dot{x}_2 = -u \end{aligned}$$

Thus, since the relative degree is  $> 1$ , we will need a higher order CBF.

Choosing  $\alpha_1(s) = \gamma_1 s$ , the higher-order CBF  $\Psi_1$  is defined as:

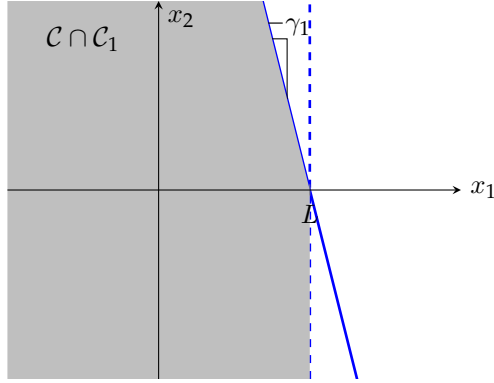
$$\begin{aligned}\Psi_1(x) &= \underbrace{L_f h(x) + L_g h(x)}_{\dot{h}} + \alpha_1(h(x)) \\ &= \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ u \end{bmatrix} + \gamma(L - x_1) \\ &= -x_2 + \gamma(L - x_1)\end{aligned}$$

We can verify that  $\Psi_1$  is a CBF by checking the condition:

$$L_g \Psi_1 = \nabla \Psi_1(x)^T g(x) = -1$$

Thus we can use  $\Psi_1(x)$  as a valid CBF. Explicitly, our safe sets are defined as:

$$\begin{aligned}\mathcal{C} &= \{x \mid h(x) \geq 0\} = \{x \mid x_1 \leq L\} \\ \mathcal{C}_1 &= \{x \mid \Psi_1(x) \geq 0\} = \{x \mid -x_2 + \gamma(L - x_1) \geq 0\}\end{aligned}$$



Explicitly, the higher-order CBF  $\Phi_1$  can be enforced via the condition (and taking  $\alpha_2(s) = \gamma_2 s$ ):

$$\begin{aligned}\Psi_1 &\geq -\alpha_2(\Psi_1(x)) \\ -\dot{x}_2 - \gamma_1 \dot{x}_1 &\geq -\gamma_2(-x_2 + \gamma_1(L - x_1)) \\ -u - \gamma_1 x_2 &\geq -\gamma_2(-x_2 + \gamma_1(L - x_1))\end{aligned}$$

**Example 3:** Let's try  $\ddot{x}_1 = u$  again, but with the safe set  $-L \leq x \leq L$ .

Choose:

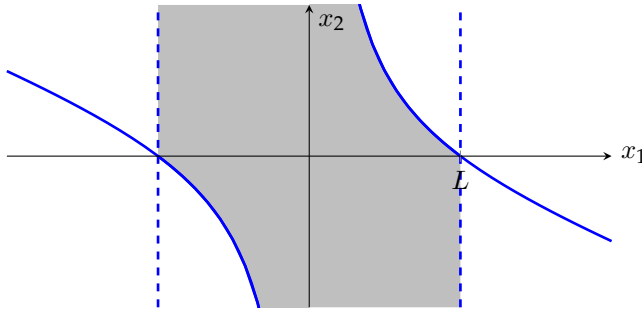
$$h(x) = \frac{1}{2}(-x_1^2 + L^2).$$

Then,

$$\begin{aligned}\dot{h}(x) &= -x_1 \dot{x}_1 = -x_1 x_2 \\ \ddot{h}(x) &= -\dot{x}_1 x_2 - x_1 \dot{x}_2 \\ &= -x_2^2 - x_1 u\end{aligned}$$

Thus the least relative degree is  $r = 2$ . This yields the higher-order CBF:

$$\Psi_1(x) = \dot{h} + \alpha_1(h(x)) = -x_1x_2 + \alpha_1\left(\frac{1}{2}(-x_1^2 + L^2)\right)$$



However, in this example, it's important to note that there are states where  $L_g\Psi_1(x) = 0$ :

$$L_g\Psi_1(x) = \nabla\Psi_1(x)^T g(x) = \begin{bmatrix} -x_2 - \gamma_1 & -x_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -x_1$$

We can also observe this by taking the derivative of  $\Psi_1$ :

$$\begin{aligned} \Psi(x) &= \dot{h} + \alpha_1(h(x)) \\ \dot{\Psi}(x) &= \ddot{h} + \alpha'_1(h(x))\dot{h}(x) \\ &= -x_2^2 - \underbrace{x_1}_{L_g\Psi_1} + \alpha'_1\left(\frac{1}{2}(-x_1^2 + L^2)\right)(-x_1x_2) \end{aligned}$$

- It is possible that  $L_gL_fh(x) = 0$ ? Yes! Whenever  $x_1 = 0$ .
- Is this a problem? We need to investigate further...

We need to see if we can find  $\alpha_2$  such that

$$\dot{\Psi}(x) + \alpha_2(\Psi(x)) \geq 0$$

whenever  $x_1 = 0$ . Evaluating at  $x_1 = 0$ :

$$\Psi(x) + \alpha_2(\Psi(x)) \big|_{x_1=0} = -x_2^2 + \alpha_2(\alpha_1(L^2/2))$$

Thus, it is always possible to find  $x_2$  large enough so that  $-x_2^2 + \alpha_2(\alpha_1(L^2/2)) < 0$ , regardless of  $\alpha_1$  and  $\alpha_2$ , so  $\Psi$  is not a valid CBF. What should we do? We have two options:

- Option 1: Nothing, except make sure  $\alpha_1$  and  $\alpha_2$  have sufficient slope so that this is only a problem when  $x_2$  is very large. This is practical, but loses theoretical guarantees

Option 2: Try a different higher-order CBF  $h$  (next example)

**Example 4:** Let's consider the same system  $\ddot{x}_1 = u$ , but we will try the control barrier function:

$$h(x) = \frac{1}{4}(-x_1^4 + L^4), \quad \dot{h}(x) = -x_1^3 x_2, \quad \ddot{h}(x) = 3x_1^2 x_2^2 - x_1^3 u.$$

Let

$$\begin{aligned} \Psi(x) &= \dot{h} + \gamma_1 h = -x_1^3 x_2 + \frac{\gamma_1}{4}(-x_1^4 + L^4) \\ \dot{\Psi}(x) &= \ddot{h} + \gamma_1 \dot{h} = -3x_1^2 x_2^2 - x_1^3 u + \frac{\gamma_1}{4}(-x_1^3 x_2) \end{aligned}$$

Then, still  $x_1^3 = 0$  whenever  $x_1 = 0$ . But,

$$L_f \Psi(x) = -3x_1^2 x_2^2 - \alpha_1 x_1^3 x_2$$

and therefore  $L_f \Psi(x) = 0$  whenever  $L_g \Psi(x) = 0$ . This means that  $\Psi$  satisfies the CBF constraint  $\sup_u L_f \Psi(x) + L_g \Psi(x)u \geq -\alpha_2(\Psi(x))$  for any  $\alpha_2$ , and  $\Psi$  is a valid CBF.

**Note:** The important takeaway is to make sure that  $L_f \Psi = 0$  whenever  $L_g \Psi = 0$ .

**Example 5:** You will implement a higher-order CBF for the cart-pole system for your homework! :)