# *ME 6402 – Lecture 19*

## Feedback Linearization 4 (Feedback Linearization for MIMO Systems)

*March 13 2025*

Overview:

- Frobenius Theorem continued
- Feedback linearization for MIMO Systems

Additional Reading:

- Khalil, Chapter 13.3

### *Full State Feedback Linearization*

Recall that our condition for $r = n$ is:

$$L_g h(x) = L_{\mathrm{ad}_f g} h(x) = \cdots = L_{\mathrm{ad}_f^{n-2} g} h(x) = 0 \text{ in a nbhd of } x_0 \qquad (1)$$

$$L_{\mathrm{ad}_f^{n-1} g} h(x_0) \neq 0. \qquad (2)$$

$L_{\mathrm{ad}_f g} h(x) = L_{[f,g]} h(x) = \frac{\partial h}{\partial x}\left(\frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)\right)$

Note: (1) can be rewritten as:

$$L_g h(x) = L_{[f,g]} h(x) = \cdots = L_{[f,[f,\ldots,[f,g]]]} h(x) = 0$$

where the benefit is that the $h(x)$ term can be moved outside:

$$\frac{\partial h}{\partial x}\begin{bmatrix} g & \mathrm{ad}_f g & \mathrm{ad}_f^2 g & \ldots & \mathrm{ad}_f^{n-2} g \end{bmatrix}$$

**Theorem**: Full-state Feedback Linearizable. *The system $\dot{x} = f(x) + g(x)u$ is feedback linearizable around $x_0$ if and only if the following two conditions hold:*

*C1)* $[g(x_0) \quad \mathrm{ad}_f g(x_0) \quad \ldots \quad \mathrm{ad}_f^{n-1} g(x_0)]$ *has rank* $n$

*C2)* $\Delta(x) = \mathrm{span}\{g(x), \mathrm{ad}_f g(x), \ldots, \mathrm{ad}_f^{n-2} g(x)\}$ *is involutive in a neighborhood of* $x_0$.

*Proof.* (if) Given C1 and C2 show that there exists $h(x)$ satisfying (1)-(2).

$\Delta(x)$ is nonsingular by C1 and involutive by C2. Thus, by the Frobenius Theorem [1], there exists $h(x)$ satisfying (1) and $dh(x) \neq 0$.

To prove (2) suppose, to the contrary, $L_{\mathrm{ad}_f^{n-1}} h(x_0) = 0$. This implies

$$dh(x_0)\underbrace{[g(x_0) \quad \mathrm{ad}_f g(x_0) \quad \ldots \quad \mathrm{ad}_f^{n-1} g(x_0)]}_{\text{nonsingular by C1}} = 0.$$

[1] Recall that the Frobenius Theorem states that a nonsingular distribution is completely integrable if and only if it is involutive. And recall that completely integrable tells us that there must exist $n-k$ functions such that $\frac{\partial \phi_i}{\partial x} f_j = 0$ and $\frac{\partial \phi_i}{\partial x}$ are linearly independent

Thus $dh(x_0) = 0$, a contradiction.

(only if) Given that $y = h(x)$ with $r = n$ exists, that is (1)-(2) hold, show that C1 and C2 are true.

We will use the following fact[2] which holds when $r = n$:

[2] see, *e.g.*, Khalil, Lemma C.8

$$L_{\mathrm{ad}_f^i g} L_f^j h(x) = \begin{cases} 0 & \text{if } i+j \leq n-2 \\ (-1)^{n-1-j} L_g L_f^{n-1} h(x) \neq 0 & \text{if } i+j = n-1. \end{cases} \quad (3)$$

Define the matrix

$$M = \begin{bmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{n-1} h \end{bmatrix} \begin{bmatrix} g & -\mathrm{ad}_f\, g & \mathrm{ad}_f^2\, g & \ldots & (-1)^{n-1}\, \mathrm{ad}_f^{n-1} g \end{bmatrix} \quad (4)$$

and note that the $(k, \ell)$ entry is:

$$\begin{aligned} M_{k\ell} &= dL_f^{k-1} h(x) (-1)^{\ell-1}\, \mathrm{ad}_f^{\ell-1}\, g(x) \\ &= (-1)^{\ell-1} L_{\mathrm{ad}_f^{\ell-1} g} L_f^{k-1} h(x). \end{aligned}$$

Then, from (3):

$$M_{k\ell} = \begin{cases} 0 & \ell + k \leq n \\ \neq 0 & \ell + k = n+1. \end{cases}$$

$$\begin{bmatrix} 0 & 0 & \cdots & \star \\ 0 & & / & \vdots \\ \vdots & \star & & \vdots \\ \star & \cdots & \cdots & \star \end{bmatrix}$$

Since the diagonal entries are nonzero, $M$ has rank $n$ and thus the factor

$$\begin{bmatrix} g & -\mathrm{ad}_f\, g & \mathrm{ad}_f^2\, g & \ldots & (-1)^{n-1}\, \mathrm{ad}_f^{n-1} g \end{bmatrix}$$

in (4) must have rank $n$ as well. Thus C1 follows.

This also implies $\Delta(x)$ is nonsingular; thus, by the Frobenius Thm,

complete integrability $\equiv$ involutivity.

$\Delta(x)$ is completely integrable since $h(x)$ satisfying (1) exists by assumption; thus, we conclude involutivity (C2). $\square$

*Example:*

Consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3 \end{aligned}$$

Feedback linearizability was shown on page 1 by inspection: $y = x_3$ gives relative degree $= 3$. Can we verify this choice of $y$ with the theorem above? We will begin by computing the elements of the span[3]:

$$f(x) = \begin{bmatrix} x_2 + 2x_1^2 \\ x_3 \\ x_1 - x_3 \end{bmatrix} \quad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$[f,g](x) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad [f,[f,g]](x) = \begin{bmatrix} 4x_1 \\ 0 \\ 1 \end{bmatrix}$$

[3] Recall that $[f,g] = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$

$$[f,g] = 0 - \begin{bmatrix} 4x_1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$
$$[f,[f,g]] = 0 - \begin{bmatrix} 4x_1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 0 \\ 1 \end{bmatrix}$$

Conditions of the theorem:

1. $\begin{bmatrix} 0 & -1 & 4x_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ full rank

2. $\Delta = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$ involutive

   $\frac{\partial h}{\partial x} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ satisfied by $h(x) = x_3$.

## *Feedback Linearization Continued*

Recall "strict feedback systems" discussed in Lecture 14:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\ &\vdots \\ \dot{x}_n &= f_n(x) + g_n(x)u. \end{aligned} \tag{5}$$

Such systems are feedback linearizable when $g_i(x_1, \ldots, x_i) \neq 0$ near the origin, $i = 1, 2, \cdots, n$, because the relative degree is $n$ with the choice of output $y = h(x) = x_1$:

$$y^{(n)} = L_f^n h(x) + \underbrace{g_1(x_1)g_2(x_1, x_2) \cdots g_n(x)}_{L_g L_f^{n-1} h(x) \neq 0} u.$$

Feedback linearizability is lost when $g_i(0) = 0$ for some $i$; however, backstepping may be applicable as the following example illustrates:

Example 1:

$$\begin{aligned}\dot{x}_1 &= x_1^2 x_2\\ \dot{x}_2 &= u.\end{aligned}$$

Treat $x_2$ as virtual control and let $\alpha_1(x_1) = -x_1$ which stabilizes the $x_1$-subsystem, as verified with Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$.
Then $z_2 := x_2 - \alpha_1(x_1)$ satisfies $\dot{z}_2 = u - \dot{\alpha}_1$, and

$$u = \dot{\alpha}_1 - \frac{\partial V_1}{\partial x_1} x_1^2 - k_2 z_2 = -x_1^2 x_2 - x_1^3 - k_2(x_2 + x_1)$$

achieves global asymptotic stability:

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \quad \Rightarrow \quad \dot{V} = -{x_1}^4 - k_2 z_2^2.$$

In contrast the system is not feedback linearizable, because condition (C1) in the theorem for feedback linearizability (Lecture 18, p.4) fails. To see this note that

$$f(x) = \begin{bmatrix} x_1^2 x_2 \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathrm{ad}_f\, g(x) = [f, g](x) = \begin{bmatrix} -x_1^2 \\ 0 \end{bmatrix},$$

thus, with $n = 2$ and $x_0 = 0$,

$$[g(x_0) \;\; \mathrm{ad}_f\, g(x_0) \;\; \ldots \;\; \mathrm{ad}_f^{n-1} g(x_0)] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

which is rank deficient.

## *Multi-Input Multi-Output Systems*

Consider now a MIMO system with $m$ inputs and $m$ outputs:

$$\begin{aligned}\dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i = f(x) + \begin{bmatrix} g_1(x) & \ldots & g_m(x) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \qquad (6)\\ y_i &= h_i(x), \quad i = 1, \cdots, m.\end{aligned}$$

Let $r_i$ denote the number of times we need to differentiate $y_i$ to hit at least one input. Then,

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \underbrace{\begin{bmatrix} L_f^{r_1} h_1(x) \\ \vdots \\ L_f^{r_m} h_m(x) \end{bmatrix}}_{=: B(x)} + \underbrace{\begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}}_{=: A(x)} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

If $A(x)$ is nonsingular, then the feedback law

$$u = A(x)^{-1}(-B(x) + v)$$

input/output linearizes the system, creating $m$ decoupled chains of integrators:

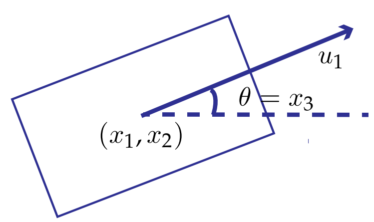
$$y_i^{(r_i)} = v_i, \quad i = 1, \dots, m.$$

We say that the system has *vector relative degree* $\{r_1, \cdots, r_m\}$ if the matrix $A(x)$ defined above is nonsingular.

Example 2: The kinematic model of a unicycle, depicted below, is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2,$$

where $u_1$ is the speed and $u_2$ is the angular velocity.




Let $y_1 = x_1$ and $y_2 = x_2$, and note that

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \end{bmatrix}}_{=: A(x)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Since $A(x)$ is singular, the system does not have a well-defined vector relative degree. □

*Normal form for MIMO systems*

The notion of zero dynamics and the normal form can be extended to MIMO systems[4]. If the system has vector relative degree $\{r_1, \cdots, r_m\}$, then $r := r_1 + \cdots + r_m \leq n$ and

[4] see, *e.g.*, Sastry, Section 9.3

$$\eta := [h_1(x)\ L_f h_1(x) \cdots L_f^{r_1 - 1} h_1(x)\ \cdots\ h_m(x)\ L_f h_m(x) \cdots L_f^{r_m - 1} h_m(x)]^T$$

defines a partial set of coordinates. As in normal form discussed in Lecture 17, one can find $n - r$ additional functions $z_1(x), \cdots, z_{n-r}(x)$ so that $x \mapsto (z, \eta)$ is a complete coordinate transformation.

Full-state feedback linearization amounts to finding $m$ output functions $h_1, \cdots, h_m$ such that the system has vector relative degree

$\{r_1, \cdots, r_m\}$ with $r_1 + \cdots + r_m = n$. Necessary and sufficient conditions for the existence of such functions, analogous to those in Lecture 18 for SISO systems, are available[5].

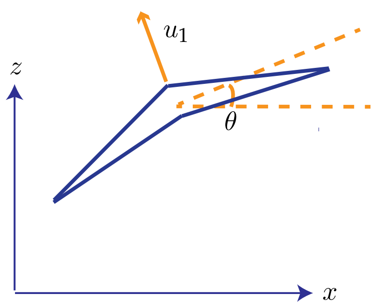
[5] see, *e.g.*, Sastry, Proposition 9.16

Example 3: Consider the following model of a *planar vertical take-off and landing* (PVTOL) aircraft[6]

[6] Sastry, Section 10.4.2

$$\begin{aligned}
\ddot{x} &= -\sin(\theta)u_1 + \mu\cos(\theta)u_2 \\
\ddot{z} &= \cos(\theta)u_1 + \mu\sin(\theta)u_2 - 1 \\
\ddot{\theta} &= u_2,
\end{aligned}$$

where $\mu$ is a constant that accounts for the coupling between the rolling moment and translational acceleration, and $-1$ in the second equation is the gravitational acceleration, normalized to unity by appropriately scaling the variables.




If we take $x$ and $z$ as the two outputs we get

$$\begin{bmatrix} \ddot{x} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \underbrace{\begin{bmatrix} -\sin\theta & \mu\cos\theta \\ \cos\theta & \mu\sin\theta \end{bmatrix}}_{A(\theta)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $A(\theta)$ is invertible when $\mu \neq 0$:

$$A^{-1}(\theta) = \begin{bmatrix} -\sin\theta & \cos\theta \\ \frac{1}{\mu}\cos\theta & \frac{1}{\mu}\sin\theta \end{bmatrix}.$$

Thus the systems has vector relative degree $\{2, 2\}$ This implies that when $\mu \neq 0$, and the input/output linearizing controller is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \frac{1}{\mu}\cos\theta & \frac{1}{\mu}\sin\theta \end{bmatrix} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right).$$

The zero dynamics is obtained by substituting $u_2^* = \frac{1}{\mu}\sin\theta$, needed to maintain $z$ at a constant value and $\dot{z}$ at zero, in the dynamical equation for $\theta$:

$$\ddot{\theta} = \frac{1}{\mu}\sin\theta.$$

The system is nonminimum phase for $\mu > 0$, since $\theta = 0$ is unstable.