ME 6402 – Lecture 1[1]

A BRIEF INTRODUCTION

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[1] Based on notes created by Murat Arcak and licensed under a

Overview

- Introduce nonlinear systems
- Define equilibria, linearization, stability in scalar systems
- Provide some canonical examples

Additional Reading:

- Khalil, Chapter 1
- Sastry, Chapter 1

Linear Systems

$$\dot{x} = Ax, \quad x(t_0) = x_0 \in \mathbb{R}^n \tag{1}$$

We use the shorthand notation $\dot{x} = f(x)$ for $\frac{d}{dt}x(t) = f(x(t))$.

Here, A is an $n \times n$ constant matrix. This linear system has the following properties:

1. Solutions always exist, and are given in closed form
$$x(t) = e^{A(t-t_0)}x_0, \ t \geq t_0$$
2. Solutions exist for all $-\infty < t < \infty$
3. Solutions are unique
4. The set of equilibrium points is the nullspace of A (i.e., connected)
5. Periodic solutions are only marginally stable, never stable (asymptotically or exponentially)

Nonlinear Systems

In comparison, nonlinear systems are more complex but also more expressive. We will consider nonlinear systems of the form:

$$\dot{x} = f(x), \ x(t_0) \in \mathbb{R}^n \tag{2}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$.

This system is time-invariant. We can also consider time-varying systems:

$\dot{x} = f(x)$	$f : \mathbb{R}^n \to \mathbb{R}^n$	time-invariant (autonomous)
$\dot{x} = f(t, x)$	$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$	time-varying (non-autonomous)

When the system has a control input $u \in \mathbb{R}^m$, the linear and nonlinear system dynamics are:

$$\dot{x} = Ax + Bu \quad \longrightarrow \quad \dot{x} = f(x, u) \tag{3}$$

Sometimes the nonlinear system can be written as $\dot{x} = f(x) + g(x)u$, which is called *control-affine* form.

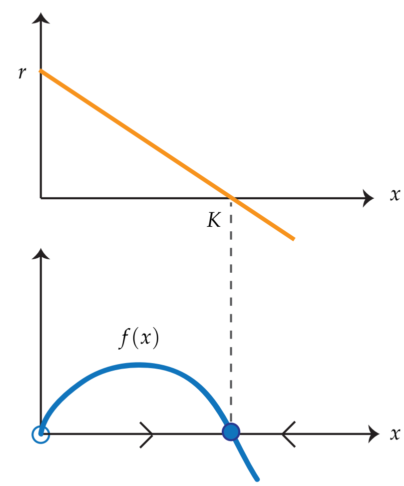
Nonlinear System Analysis and Design

- Analysis (first half of course): Determine stability, convergence, etc of $\dot{x} = f(x)$
- Design (second half of course): Choose u as a function of x to achieve desired behavior

Motivating Scalar Example

Logistic growth model in population dynamics

$$\dot{x} = f(x) = \underbrace{r\left(1 - \frac{x}{K}\right)}_{\text{growth rate}} x, \qquad r > 0, \;\; K > 0 \tag{4}$$



$x > 0$ denotes the population, K is called the carrying capacity, and r is the intrinsic growth rate.

For systems with a scalar state variable $x \in \mathbb{R}$, stability can be determined from the sign of $f(x)$ around the equilibrium. In this example $f(x) > 0$ for $x \in (0, K)$, and $f(x) < 0$ for $x > K$; therefore

$$\begin{aligned} x = 0 &\quad \text{unstable equilibrium} \\ x = K &\quad \text{asymptotically stable.} \end{aligned}$$

In general, $x = x^*$ is an equilibrium for $\dot{x} = f(x)$ if $f(x^*) = 0$

Linearization

Local stability properties of x^* can be determined by linearizing the vector field $f(x)$ at x^*. These linearized dynamics are expressed in terms of deviations from the equilibrium $\tilde{x} = x - x^*$. The dynamics of $\tilde{x}$ are given by:

$$\dot{\tilde{x}} \triangleq f(x^* + \tilde{x}) \tag{5}$$

The linearization of these dynamics can be solved as before, using a first-order Taylor series approximation:

Note this comes from the standard first-order Taylor series approximation: $f(x) \approx f(x^*) + f'(x^*)(x - x^*)$ and substituting in $x = x^* + \tilde{x}$

$$f(x^* + \tilde{x}) = \underbrace{f(x^*)}_{=\,0} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=x^*}}_{\triangleq\,A} \tilde{x} + \text{higher order terms} \tag{6}$$

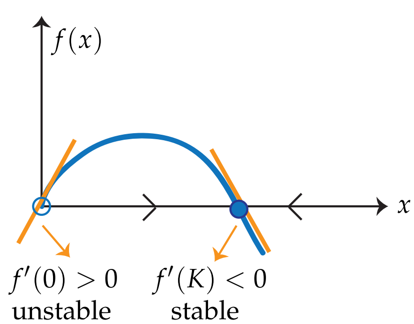
for $\tilde{x} = x - x^*$ Thus, the linearized model is:

$$\dot{\tilde{x}} = A\tilde{x}. \tag{7}$$

If $\Re\lambda_i(A) < 0$ for each eigenvalue λ_i of A, then x^* is asymp. stable.

If $\Re\lambda_i(A) > 0$ for some eigenvalue λ_i of A, then x^* is unstable.

Example: Logistic growth model above:



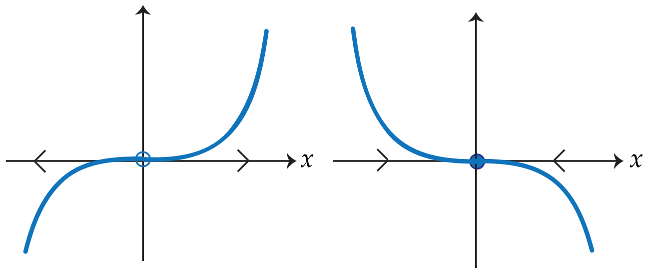
Caveats:

1. Only local properties can be determined from the linearization.

   Example: The logistic growth model linearized at $x = 0$ ($\dot{x} = rx$) would incorrectly predict unbounded growth of $x(t)$. In reality, $x(t) \to K$.

2. If $\Re\lambda_i(A) \leq 0$ with equality for some i, then linearization is inconclusive as a stability test. Higher order terms determine stability.

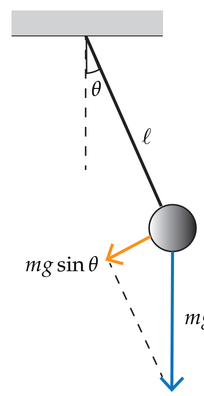
Example: $f(x) = x^3$ vs. $f(x) = -x^3$



$f'(0) = 0$ in each case, but one is stable and the other is unstable.

Motivating Example 2

Let's consider the pendulum system with a frictional force resisting the motion (coefficient of friction k):



The damping torque acting on the pendulum is $-\ell(k\ell\theta)$ for the planar pendulum.

$$\ell m\ddot{\theta} = -k\ell\dot{\theta} - mg\sin\theta \tag{8}$$

or

$$\ddot{\theta} = \frac{-k}{m}\dot{\theta} - \frac{g}{l}\sin\theta \tag{9}$$

Note: These dynamics can be derived from the Lagrangian:

$$\begin{aligned}\mathcal{L}(\theta,\dot{\theta}) &= KE - PE \\ &= \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta\end{aligned}$$

with the equations of motion given via the Euler-Lagrange equations (d'Alembert Principle):

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} &= \tau_{ext} \\ \frac{d}{dt}\left(m\ell^2\dot{\theta}\right) + mg\ell\sin\theta &= -k\ell^2\dot{\theta} \\ m\ell^2\ddot{\theta} + mg\ell\sin\theta &= -k\ell^2\dot{\theta} \\ \ddot{\theta} + \frac{g}{\ell}\sin\theta &= -\frac{k}{m}\dot{\theta} \\ \ddot{\theta} &= -\frac{k}{m}\dot{\theta} - \frac{g}{\ell}\sin\theta\end{aligned}$$

Define $x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$. State space: $S^1 \times \mathbb{R}$.

The system dynamics $\dot{x}$ can be rewritten in terms of this state as:

$$\dot{x} = \begin{bmatrix}\dot{\theta} \\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{\theta} \\ -\frac{k}{m}\dot{\theta} - \frac{g}{\ell}\sin\theta\end{bmatrix} = \begin{bmatrix}x_2 \\ -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1\end{bmatrix} \tag{10}$$

Equilibria: $(0,0)$ and $(\pi,0)$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & -\frac{k}{m} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} & \text{(stable) at } x_1 = 0 \\ \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} & \text{(unstable) at } x_1 = \pi \end{cases}$$

Phase portrait: plot of $x_1(t)$ vs. $x_2(t)$ for 2nd order systems

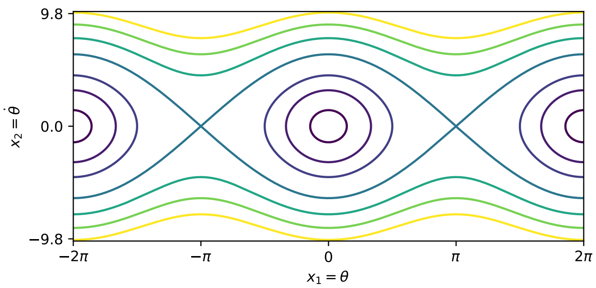


Figure 1: Phase portrait of the pendulum for the undamped case $k = 0$ with $m = 1$, $g = 9.8$, $\ell = 1$.