

Topics Covered:

- Trajectory Design
- Cubic Polynomials / Splines
- Manipulator Trajectories

Additional Reading:

- LP Chapter 9
- Craig Chapter 7

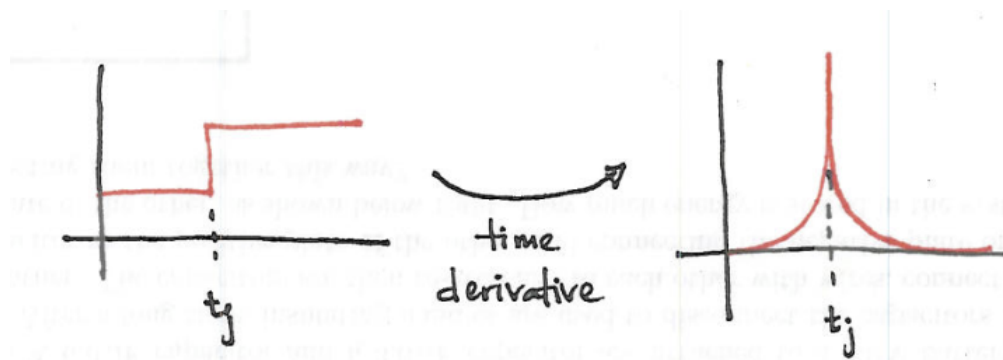
Introduction to Trajectory Design

Goal: to design realistic/feasible trajectories connecting different configurations

- joint configurations
- group configurations
- both levels? → sometimes not possible using only one configuration space. may need to utilize both.

Trajectory Design

- want sufficiently smooth trajectories (continuous derivatives). No jerky motions which cause wear, induce vibrations, excite resonances, etc. The delta function has an infinite derivative

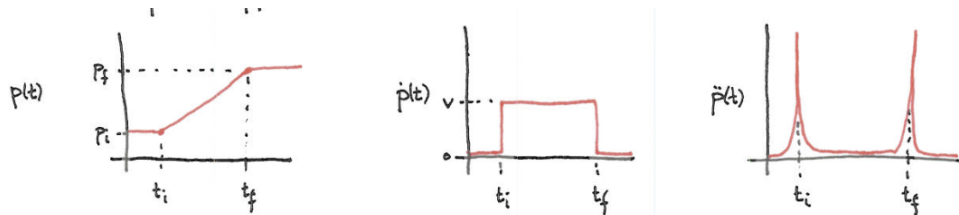


at t_j . In frequency domain, excites all frequencies.

- some trajectories may require intermediate points between the initial and final configurations. (known as: way points, via points, knot points)

How should trajectories be generated/designed?

- The simplest approach is linear



- need to manage velocity better
- let's add velocity constraints on initial and final configurations

Cubic Polynomials / Splines

A polynomial is defined as:

$$p(t) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Why cubic? Because of the number of constraints we are imposing

- | | | |
|----------------------------------|---------------------|-------------|
| 4 constraints on the polynomial: | 1. initial point | p_i |
| | 2. final point | p_f |
| | 3. initial velocity | $\dot{p}_i$ |
| | 4. final velocity | $\dot{p}_f$ |

So, minimally, a cubic polynomial is required.

To simply connect two points, constraints are

$$\begin{aligned} p(0) &= p_i, & \dot{p}(0) &= 0 \\ p(t_f) &= p_f, & \dot{p}(t_f) &= 0 \end{aligned}$$

Note:

$$\begin{aligned} \dot{p}(t) &= a_1 + 2a_2t + 3a_3t^2 \\ \ddot{p}(t) &= 2a_2 + 6a_3t \end{aligned}$$

Thus, we can solve for the polynomial coefficients:

$$\begin{aligned}
 p(0) &= a_0 & &= p_i \\
 p(t_f) &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 = p_f & &= p_f \\
 \dot{p}(0) &= a_1 & &= 0 \\
 \dot{p}(t_f) &= a_1 + 2a_2 t_f + 3a_3 t_f^2 = 0 & &= 0
 \end{aligned}$$

Putting these equations into matrix form yields:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix}}_{A(t_f)} \vec{a} = \vec{p}_0, \quad \text{where } \vec{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \vec{p}_0 = \begin{bmatrix} p_i \\ p_f \\ 0 \\ 0 \end{bmatrix}$$

By inverting this matrix, we can directly obtain a matrix form of the polynomial coefficients:

$$\vec{a} = P(t_f) \vec{p}_0, \quad \text{where } P(t_f) = A^{-1}(t_f)$$

$$P(t_f) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3/t_f^2 & 3/t_f^2 & -2/t_f & -1/t_f \\ 2/t_f^3 & -2/t_f^3 & 1/t_f^2 & 1/t_f^2 \end{bmatrix}$$

but, notice that $\vec{p}_0$ has zeros in the last two rows, thus the last two columns of $P(t_f)$ are not needed.

$$\vec{a} = P_{simp}(t_f) \begin{Bmatrix} p_i \\ p_f \end{Bmatrix}, \quad \text{where } P_{simp}(t_f) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -3/t_f^2 & 3/t_f^2 \\ 2/t_f^3 & -2/t_f^3 \end{bmatrix}$$

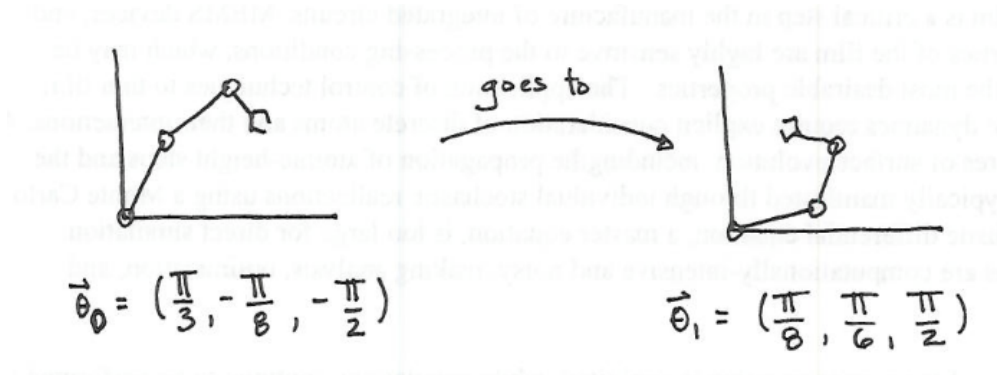
worked out individually, this is

$$\begin{aligned}
 a_0 &= p_i & a_2 &= \frac{3}{t_f^2}(p_f - p_i) \\
 a_1 &= 0 & a_3 &= \frac{2}{t_f^3}(p_i - p_f)
 \end{aligned}$$

What value should t_f take?

- certainly should be short enough so that trajectory doesn't take forever.
- not too short though: actuator limits, end-effector velocity/acceleration limits
- have decent initial guess based on limits, $v_{lim} \cdot t = d$ (v_{lim} is the limit/nominal velocity, t is the unknown time to solve, and d is the distance)

Example



Goal: join initial and final joint configurations with trajectory whose velocity does not exceed $\pi/4$ rad/sec.

$$t_f = \frac{d}{v_{max}} = \frac{\pi \text{ rad}}{\pi/4 \text{ rad/sec}} = 4 \text{ sec}$$

We can solve for each joint angle polynomial by plugging the values for $\theta_i(0)$ and $\theta_i(t_f)$ into the equation we obtained ($\theta_i(t) = p_i + 0t + \frac{3}{t_f^2}(p_f - p_i)t^2 + \frac{2}{t_f^3}(p_i - p_f)t^3$):

$$\begin{aligned} \theta_1(t) &= \frac{\pi}{3} + \frac{3}{4^2} \left(\frac{\pi}{8} - \frac{\pi}{3} \right) t^2 + \frac{2}{4^3} \left(\frac{\pi}{3} - \frac{\pi}{8} \right) t^3 \\ &= \frac{\pi}{3} + \frac{3}{16} \frac{-5\pi}{24} t^2 + \frac{2}{64} \frac{5\pi}{24} t^3 \\ &= \frac{\pi}{3} - \frac{5\pi}{128} t^2 + \frac{5\pi}{768} t^3 \\ \theta_2(t) &= -\frac{\pi}{8} + \frac{3}{4^2} \left(\frac{\pi}{6} + \frac{\pi}{8} \right) t^2 + \frac{2}{4^3} \left(-\frac{\pi}{8} - \frac{\pi}{6} \right) t^3 \\ &= -\frac{\pi}{8} + \frac{3}{16} \frac{7\pi}{24} t^2 + \frac{2}{64} \frac{-7\pi}{24} t^3 \\ &= -\frac{\pi}{8} + \frac{7\pi}{128} t^2 - \frac{7\pi}{768} t^3 \\ \theta_3(t) &= -\frac{\pi}{2} + \frac{3}{4^2} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) t^2 + \frac{2}{4^3} \left(-\frac{\pi}{2} - \frac{\pi}{2} \right) t^3 \\ &= -\frac{\pi}{2} + \frac{3\pi}{16} t^2 - \frac{\pi}{32} t^3 \end{aligned}$$

where each of these polynomials are enforced for $t \in [0, 4]$.

If we examine $\dot{\theta}_3(t)$, it actually goes more than $\pi/4$ rad/sec (due to initial acceleration and final deceleration).

Extending t_f to 7 seconds reduces the velocity below $\pi/4$ rad/sec:

$$\begin{aligned}\theta_1(t) &= \frac{\pi}{3} - \frac{5\pi}{392}t^2 + \frac{5\pi}{4116}t^3 \\ \theta_2(t) &= -\frac{\pi}{8} + \frac{\pi}{5}t^2 - \frac{\pi}{588}t^3 \\ \theta_3(t) &= -\frac{\pi}{2} + \frac{3\pi}{49}t^2 - \frac{2\pi}{343}t^3\end{aligned}$$

where now the polynomials are for $t \in [0, 7]$.