Topics Covered:

- Review
- Derivation for Product of Homogeneous Transformation Matrices
- Derivation for Product of Exponentials

Additional Reading:

- LP Chapter 5; MLS Chapter 3, Section 4

Review

Last lecture, we introduced the manipulator jacobian which maps joint velocities to the Lie algebra of the end-effector frame relative to either the fixed frame (spatial frame) or the end-effector frame (body frame):

$$
\begin{aligned}
\hat{\xi}^s &= J^s(\theta)\dot{\theta} \\
&= \begin{bmatrix} \xi_1 & \xi_2' & \cdots & \xi_n' \end{bmatrix} \dot{\theta} \\
&= \begin{bmatrix} \left(\frac{\partial g_e}{\partial \theta_1} g_e^{-1}\right)^\vee & \cdots & \left(\frac{\partial g_e}{\partial \theta_n} g_e^{-1}\right)^\vee \end{bmatrix} \dot{\theta} \\
&= \begin{bmatrix} \xi_1 & \mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}} \xi_2 & \cdots & \mathrm{Ad}_{\prod_{j=1}^{n-1} e^{\hat{\xi}_j\theta_j}} \xi_n \end{bmatrix} \dot{\theta} \\
\hat{\xi}^b &= J^b(\theta)\dot{\theta} \\
&= \begin{bmatrix} \xi_1^\dagger & \cdots & \xi_n^\dagger \end{bmatrix} \dot{\theta} \\
&= \begin{bmatrix} \left(g_e^{-1} \frac{\partial g_e}{\partial \theta_1}\right)^\vee & \cdots & \left(g_e^{-1} \frac{\partial g_e}{\partial \theta_n}\right)^\vee \end{bmatrix} \dot{\theta} \\
&= \begin{bmatrix} \mathrm{Ad}^{-1}_{\prod_{j=1}^{n} e^{\hat{\xi}_j\theta_j} g_0} \xi_1 & \cdots & \mathrm{Ad}^{-1}_{e^{\hat{\xi}_n\theta_n} g_0} \xi_n \end{bmatrix} \dot{\theta}
\end{aligned}
$$

In the rest of today's lecture we will derive these expressions, as well as slightly different expressions that use the following "body terms":

$$
\begin{aligned}
J_i^b(\theta_i) &= g_i^{-1} \frac{\partial g_i}{\partial \theta_i}, \\
J_i^s(\theta_i) &= \mathrm{Ad}_{g_i} J_i^b(\theta_i) = \frac{\partial g_i}{\partial \theta_i} g_i^{-1}.
\end{aligned}
$$

The advantage of these terms is that the partial derivatives are easier to obtain in practice.

Uses of Manipulator Jacobians

The manipulator Jacobian is a key tool in robotics for a variety of applications including inverse kinematics, path planning, force control, and singularity/workspace analysis. The key equations that are used with Jacobians are the following.

For example, towards inverse kinematics, the Newton-Raphson method uses the iterative update law:

$$\theta_{k+1} = \theta_k + \underbrace{(\frac{\partial g}{\partial \theta}(\theta_k))^{-1}(g(\theta_k))}_{\delta(\theta_k)} \qquad (g(\theta) := x_d - f(\theta))$$

$$\delta(\theta) = J^{-1}(\theta_0)(x_d - f(\theta_0))$$

Here, if $J \in \mathbb{R}^{m \times n}$ is not square (or if we are passing through singularities), we will use the pseudo-inverse:

$$J^{-1} \approx J^\dagger := \begin{cases} J^T(JJ^T)^{-1} & \text{if } J \text{ is fat } (n > m) \\ (J^TJ)^{-1}J^T & \text{if } J \text{ is tall } (n < m) \end{cases}$$

The process of solving for joint velocities is even simpler with the Jacobian:

$$\dot{\theta} = J^{-1}(\theta)\xi_{des}$$

Formulas of the Manipulator Jacobian using Individual Body Terms

In the rest of lecture, we will derive the following expressions in terms of the "body terms". These body terms are defined as:

$$J_i^b = g_i^{-1}\frac{\partial g_i}{\partial \theta_i},$$
$$J_i^s = \mathrm{Ad}_{g_i} J_i^b = \frac{\partial g_i}{\partial \theta_i} g_i^{-1}.$$

There will be two sets of formulas depending on if you are using the Product of Lie Groups or Product of Exponentials formulation for the forward kinematics.

Using the Product of Exponentials, we have:

$$J^b(\theta) = \left[\mathrm{Ad}^{-1}_{\prod_{j=2}^n \left(e^{\xi_j \theta_j}\right) g_0} J_1^b \quad \cdots \quad \mathrm{Ad}^{-1}_{\prod_{k+1}^n \left(e^{\xi_j \theta_j}\right) g_0} J_k^b \quad \cdots \quad \mathrm{Ad}^{-1}_{g_0} J_n^b\right]$$
$$J^s(\theta) = \left[J_1^s(\theta) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{k-1} e^{\xi_j \theta_j}} J_k^s(\theta_k) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{n-1} e^{\xi_j \theta_j}} J_n^s(\theta)\right]$$

Using the Product of Lie Groups, we have:

$$J^b(\theta) = \left[\mathrm{Ad}^{-1}_{\left(\prod_{j=2}^{n+1} g_j\right)} J_1^b(\theta_1) \quad \cdots \quad \mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{n+1} g_j\right)} J_k^b(\theta_k) \quad \cdots \quad \mathrm{Ad}^{-1}_{g_{n+1}} J_n^b(\theta_n)\right]$$
$$J^s(\theta) = \left[J_1^s(\theta_1) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{k-1} g_j} J_k^s(\theta_k) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{n-1} g_j} J_n^s(\theta_n)\right]$$

The benefit here is that the computations for each body term are relatively easy to compute since they only depend on a single joint and link. For example, returning to our previous example:

$$g_1(\theta_1) = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$g_2(\theta_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_2) & \sin(\theta_2) & 0 \\ 0 & -\sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$J_2^s = \frac{\partial g_2}{\partial \theta_2} g_2^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & -\cos(\theta_2) & -\sin(\theta_2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_2) & -\sin(\theta_2) & 0 \\ 0 & \sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J_2^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin(\theta_2)\cos(\theta_2) + \cos(\theta_2)\sin(\theta_2) & \sin^2(\theta_2) + \cos^2(\theta_2) & 0 \\ 0 & -\cos(\theta_2)\cos(\theta_2) - \sin(\theta_2)\sin(\theta_2) & +\sin(\theta_2)\cos(\theta_2) - \cos(\theta_2)\sin(\theta_2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the formula for J_2^s is then:

$$\begin{aligned}
(J_2^s(\theta))^\wedge &= \mathrm{Ad}_{g_1(\theta_1)} J_2^s(\theta_2) \\
&= g_1 J_2^s g_1^{-1} \\
&= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & -l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -\sin(\theta_1) & 0 \\ 0 & 0 & \cos(\theta_1) & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & -l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -\sin(\theta_1) & l_0 \sin(\theta_1) \\ 0 & 0 & \cos(\theta_1) & -l_0 \cos(\theta_1) \\ \sin(\theta_1) & -\cos(\theta_1) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
J_2^s(\theta) &= \begin{bmatrix} l_0 \sin(\theta_1) \\ -l_0 \cos(\theta_1) \\ 0 \\ -\cos(\theta_1) \\ -\sin(\theta_1) \\ 0 \end{bmatrix}
\end{aligned}$$

Notice that this would match what we had before:

$$J_2^s = \begin{bmatrix} -\omega_2' \times q_1 \\ \omega_2' \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} -\cos(\theta_1) \\ -\sin(\theta_1) \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} l_0 \sin(\theta_1) \\ -l_0 \cos(\theta_1) \\ 0 \\ -\cos(\theta_1) \\ -\sin(\theta_1) \\ 0 \end{bmatrix}$$

Derivation using Product of Lie Groups

Let $\theta = \theta(t)$ be the joint configuration of a manipulator. The end-effector trajectory is then $g_e(\theta(t))$.

Taking the time derivative, we get:

$$\begin{aligned}
\dot{g}_e &= \frac{d}{dt}\left[g_e(\theta(t))\right] \\
&= \frac{\partial g_e}{\partial \theta}\frac{\partial \theta}{\partial t} \\
&= Dg_e(\theta(t))\dot{\theta}(t) \\
&= \begin{bmatrix} | & & | \\ \frac{\partial g_e}{\partial \theta_1} & \dots & \frac{\partial g_e}{\partial \theta_n} \\ | & & | \end{bmatrix} \dot{\theta}(t)
\end{aligned}$$

Let's think about what these partial derivatives ($\frac{\partial g_e}{\partial \theta_i}$) mean when the representation for g_e is in homogeneous coordinates?

First, let's switch frames since $\dot{g}_e = Dg_e\dot{\theta}$ has mixed frames.

- Body Frame: $J^b(\theta) = g_e^{-1}(\theta) \cdot \frac{\partial g_e(\theta)}{\partial \theta}$ (this is the body manipulator Jacobian)
- Spatial Frame: $J^s(\theta) = \frac{\partial g_e(\theta)}{\partial \theta} \cdot g_e^{-1}(\theta)$ (this is the spatial manipulator Jacobian)

In mixed frames, $J(\theta) = \frac{\partial g_e(\theta)}{\partial \theta}$ is called the *manipulator Jacobian*.

In practice, we work in homogeneous form where needed or when the computations are simpler, then convert to vector form by unhatting the result.

To do this derivation, let's consider a basic example that only has 2 joints and 2 links, with the joints located at the start of the first and second link:

$$g_e(\theta) = g_1(\theta_1)g_2(\theta_2)g_3$$

To compute the body manipulator Jacobian, we would do:

$$
\begin{aligned}
J^b(\theta) &= g_e^{-1}(\theta)\cdot\frac{\partial g_e(\theta)}{\partial\theta} \\
&= g_e^{-1}(\theta)\begin{bmatrix} | & | \\ \frac{\partial g_e}{\partial\theta_1} & \frac{\partial g_e}{\partial\theta_2} \\ | & | \end{bmatrix} \\
&= g_e^{-1}(\theta)\begin{bmatrix} \frac{\partial g_1}{\partial\theta_1} g_2 g_3 & g_1 \frac{\partial g_e}{\partial\theta_2} g_3 \end{bmatrix} \\
&= g_3^{-1} g_2^{-1} g_1^{-1}\begin{bmatrix} \frac{\partial g_1}{\partial\theta_1} g_2 g_3 & g_1 \frac{\partial g_e}{\partial\theta_2} g_3 \end{bmatrix} \\
&= \begin{bmatrix} g_3^{-1} g_2^{-1} \underbrace{g_1^{-1}\frac{\partial g_1}{\partial\theta_1}}_{\text{body term}} g_2 g_3 & g_3^{-1} \underbrace{g_2^{-1}\cancel{g_1^{-1}} g_1 \frac{\partial g_e}{\partial\theta_2}}_{\text{body term}} g_3 \end{bmatrix}
\end{aligned}
$$

Let's look at these two body terms in particular:

$$
J_1^b = g_1^{-1}\frac{\partial g_1}{\partial\theta_1} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\sin(\theta_1) & -\cos(\theta_1) & 0 \\ \cos(\theta_1) & -\sin(\theta_1) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
J_2^b = g_2^{-1}\frac{\partial g_2}{\partial\theta_2} = \begin{bmatrix} \cos(\theta_2) & \sin(\theta_2) & -\cos(\theta_2) l_1 \\ -\sin(\theta_2) & \cos(\theta_2) & -\sin(\theta_2) l_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\sin(\theta_2) & -\cos(\theta_2) & 0 \\ \cos(\theta_2) & -\sin(\theta_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Plugging these terms back in:

$$
\begin{aligned}
J^b(\theta) &= \begin{bmatrix} g_3^{-1} g_2^{-1} J_1^b g_2 g_3 & g_3^{-1} J_2^b g_3 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Ad}_{g_3^{-1}} \mathrm{Ad}_{g_2^{-1}} J_1^b & \mathrm{Ad}_{g_3^{-1}} J_2^b \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Ad}_{g_3^{-1} g_2^{-1}} J_1^b & \mathrm{Ad}_{g_3^{-1}} J_2^b \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Ad}_{(g_2 g_3)^{-1}} J_1^b & \mathrm{Ad}_{g_3^{-1}} J_2^b \end{bmatrix}
\end{aligned}
$$

This can be written as the general expression:

$$
J^b(\theta) = \begin{bmatrix} \mathrm{Ad}^{-1}_{\left(\prod_{j=2}^{n+1} g_j\right)} J_1^b(\theta_1) & \cdots & \mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{n+1} g_j\right)} J_k^b(\theta_k) & \cdots & \mathrm{Ad}^{-1}_{g_{n+1}} J_n^b(\theta_n) \end{bmatrix}
$$

In the example, we would get the body velocity of the end-effector as a function of joint velocities $\dot{\theta}$ using:

$$
\begin{aligned}
\xi^b &= J^b(\theta)\dot{\theta} \\
\xi^b &= \mathrm{Ad}_{(g_2 g_3)^{-1}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \dot{\theta}_1 + \mathrm{Ad}_{g_3^{-1}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \dot{\theta}_2
\end{aligned}
$$

Important note: Order of implementation is irrelevant here, $(\text{Ad}_h\hat{\xi})^\vee = \text{Ad}_h(\hat{\xi}^\vee)$. Remember that $\text{Ad}_h\hat{\xi} = h\hat{\xi}h^{-1}$. So here, $(\text{Ad}_h\hat{\xi})^\vee = (h\hat{\xi}h^{-1})^\vee = \text{Ad}_h\xi$. Also, Ad_h can be written as a special matrix as defined in the following lemma.

Lemma 2.13 from MLS:

Lemma. *If $\hat{\xi} \in \mathfrak{se}(3)$ is a twist with twist coordinates $\xi \in \mathbb{R}^6$, then for any $g \in SE(3)$, $g\hat{\xi}g^{-1}$ is a twist with twist coordinates $Ad_g\xi \in \mathbb{R}^6$. This adjoint is defined as:*

$$Ad_g = \begin{bmatrix} R & [d]_\times R \\ 0 & R \end{bmatrix}$$

We can derive this adjoint matrix as follows:
Let $\xi^b = \begin{bmatrix} v^b \\ \omega^b \end{bmatrix}$ and $g_{sb} = \begin{bmatrix} R_{sb} & d_{sb} \\ 0 & 1 \end{bmatrix}$. Then, we can write:

$$\begin{aligned} \omega^s &= R_{sb}\omega^b \\ v^s &= -\omega^s \times d_{sb} + \dot{d}_{sb} \\ &= d_{sb} \times (R_{sb}\omega^b) + R_{sb}v^b \qquad \text{(Since } v^b = R_{sb}^T\dot{d}_{sb}) \end{aligned}$$

$$\begin{bmatrix} v^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} R_{sb} & [d_{sb}]_\times R_{sb} \\ 0 & R_{sb} \end{bmatrix} \begin{bmatrix} v^b \\ \omega^b \end{bmatrix}$$

This adjoint matrix transforms twists from one coordinate frame to another.

Continuing with our example, to get the spatial Jacobian, we can use the adjoint of g_e with the body Jacobian:

$$J^s(\theta) = \text{Ad}_{g_e} J^b(\theta)$$

However, as you may notice, we will now have a combination of adjoint terms. Specifically, the kth column of the manipulator Jacobian is:

$$\text{Ad}_{g_e(\theta)}\text{Ad}^{-1}_{\left(\prod_{j=k+1}^{3} g_j\right)} J_k^b$$

Looking at just the adjoint terms:

$$\text{Ad}_{g_e(\theta)}\text{Ad}^{-1}_{\left(\prod_{j=k+1}^{3} g_j\right)} = \text{Ad}_{\left(\prod_{j=1}^{3} g_j\right)}\text{Ad}^{-1}_{\left(\prod_{j=k+1}^{3} g_j\right)}$$

We can also rearrange the right-hand term by observing:

$$\begin{aligned} \text{Ad}^{-1}_{(g_1g_2g_3)}\xi &= (g_1g_2g_3)^{-1}\,\xi\,(g_1g_2g_3) \\ &= g_3^{-1}g_2^{-1}g_1^{-1}\xi\left(g_3^{-1}g_2^{-1}g_1^{-1}\right)^{-1} \\ &= \text{Ad}_{\left(g_3^{-1}g_2^{-1}g_1^{-1}\right)} \end{aligned}$$

So using this pattern with our previous adjoint terms:

$$\mathrm{Ad}_{\left(\prod_{j=1}^{3} g_j\right)}\mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{3} g_j\right)} = \mathrm{Ad}_{\left(\prod_{j=1}^{3} g_j\right)}\mathrm{Ad}_{\left(\prod_{j=3}^{k+1} g_j^{-1}\right)}$$

However, when we actually apply these adjoints, we would end up seeing a lot of terms cancel:

$$\begin{aligned}
\mathrm{Ad}_{\left(\prod_{j=1}^{3} g_j\right)}\mathrm{Ad}_{\left(\prod_{j=3}^{k+1} g_j^{-1}\right)} J_k^b &= (g_1 g_2 \cdots g_3)\left(g_3^{-1}\cdots g_{k+1}^{-1}\right) J_k^b \left(g_3^{-1}\cdots g_{k+1}^{-1}\right)^{-1}(g_1\cdots g_3)^{-1} \\
&= (g_1 g_2 \cdots g_3)\left(g_3^{-1}\cdots g_{k+1}^{-1}\right) J_k^b \left(g_{k+1}\cdots g_3\right)\left(g_3^{-1}\cdots g_1^{-1}\right) \\
&= (g_1\cdots g_k) J_k^b (g_k^{-1}\cdots g_1^{-1}) \\
&= \mathrm{Ad}_{\left(\prod_{j=1}^{k} g_j\right)} J_k^b
\end{aligned}$$

Thus, plugging this back into our expression for the spatial Jacobian:

$$\begin{aligned}
J^s(\theta) &= \mathrm{Ad}_{g_e(\theta)} J^b(\theta) \\
&= \begin{bmatrix} | & & | & & | \\ \mathrm{Ad}_{g_1} J_1^b & \cdots & \mathrm{Ad}_{\prod_{j=1}^{k} g_j} J_k^b & \cdots & \mathrm{Ad}_{\prod_{j=1}^{n} g_j} J_n^b \\ | & & | & & | \end{bmatrix}
\end{aligned}$$

Lastly, since we want everything in terms of the spatial frame, we can substitute in:

$$J_i^s = \mathrm{Ad}_{g_i} J_i^b$$

by pulling out the last term of the product series in the previous expression:

$$\begin{aligned}
J^s(\theta) &= \begin{bmatrix} \mathrm{Ad}_{g_1} J_1^b & \cdots & \mathrm{Ad}_{\prod_{j=1}^{k-1} g_j}\mathrm{Ad}_{g_k} J_k^b & \cdots & \mathrm{Ad}_{\prod_{j=1}^{n-1} g_j}\mathrm{Ad}_{g_n} J_n^b \end{bmatrix} \\
&= \begin{bmatrix} J_1^s(\theta_1) & \cdots & \mathrm{Ad}_{\prod_{j=1}^{k-1} g_j} J_k^s(\theta_k) & \cdots & \mathrm{Ad}_{\prod_{j=1}^{n-1} g_j} J_n^s(\theta_n) \end{bmatrix}
\end{aligned}$$

Derivation using Product of Exponentials

We can repeat a very similar process for the Product of Exponentials method of forward kinematics by representing our end-effector configuration as:

$$g_e = e^{\xi_1\theta_1}\cdots e^{\xi_n\theta_n} g_0,$$

and plugging this into the formula for the body manipulator jacobian ($J^b(\theta) = g_e^{-1}\frac{\partial g_e(\theta)}{\partial\theta}$),

$$\begin{aligned}
J^b(\theta) &= \left(e^{\xi_1\theta_1}\cdots e^{\xi_n\theta_n} g_0\right)^{-1}\left[\frac{\partial\left(e^{\xi_1\theta_1}\cdots e^{\xi_n\theta_n} g_0\right)}{\partial\theta_1}\cdots\frac{\partial\left(e^{\xi_1\theta_1}\cdots e^{\xi_n\theta_n} g_0\right)}{\partial\theta_n}\right] \\
&= \left(e^{\xi_1\theta_1}\cdots e^{\xi_n\theta_n} g_0\right)^{-1}\left[\xi_1 e^{\xi_1\theta_1}\cdots e^{\xi_n\theta_n} g_0 \cdots \xi_n e^{\xi_1\theta_1}\cdots e^{\xi_n\theta_n} g_0\right] \\
&= \left(g_0^{-1} e^{-\xi_n\theta_n}\cdots e^{-\xi_1\theta_1}\right)\left[\xi_1 e^{\xi_1\theta_1}\cdots e^{\xi_n\theta_n} g_0 \cdots \left(e^{\xi_1\theta_1}\cdots \xi_n e^{\xi_n\theta_n}\right) g_0\right]
\end{aligned}$$

Here, we can further simplify these terms by substituting in the body terms:

$$\begin{aligned} J_i^b &= g_i^{-1}\frac{\partial g_i}{\partial \theta_i} \\ &= e^{-\xi_i\theta_i}\xi_i e^{\xi_i\theta_i} = J_i^b. \end{aligned}$$

$$\begin{aligned} J^b(\theta) &= \left[\left(g_0^{-1}e^{-\xi_n\theta_n}\cdots e^{-\xi_2\theta_2}\right)J_1^b\left(e^{\xi_2\theta_2}\cdots e^{\xi_n\theta_n}g_0\right) \quad \cdots \quad g_0^{-1}J_n^b g_0\right] \\ &= \left[\mathrm{Ad}^{-1}_{e^{\xi_2\theta_2}\cdots e^{\xi_n\theta_n}g_0}J_1^b \quad \cdots \quad \mathrm{Ad}^{-1}_{g_0}J_n^b\right] \\ &= \left[\mathrm{Ad}^{-1}_{\prod_{j=2}^{n}\left(e^{\xi_j\theta_j}\right)g_0}J_1^b \quad \cdots \quad \mathrm{Ad}^{-1}_{\prod_{k+1}^{n}\left(e^{\xi_j\theta_j}\right)g_0}J_k^b \quad \cdots \quad \mathrm{Ad}^{-1}_{g_0}J_n^b\right] \end{aligned}$$

Lastly, we can similarly convert this to the spatial Jacobian:

$$J^s(\theta) = \left[J_1^s(\theta) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{k-1}e^{\xi_j\theta_j}}J_k^s(\theta_k) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{n-1}e^{\xi_j\theta_j}}J_n^s(\theta)\right]$$