Topics Covered:

- Exponential Representation of Group Motion / Displacements for $SE(3)$ and $se(3)$
- Exponential Representation of Group Motion / Displacements for $SE(2)$ and $se(2)$
- Notes on Rotations in $SE(3)$

Additional Reading:

- MLS Chapter 2, Section 3.3; LP 3.3.3

Recall from last class that we discussed the exponential representation of group motion/displacements for $SO(3)$ and $so(3)$. This allows us to compute $R \in SO(3)$ using the Lie-algebra element ($\omega \in \mathfrak{so}(3)$) using Rodrigues' formula:

$$e^{[\omega]_\times \tau} = I + \frac{[\omega]_\times}{\|\omega\|}\sin(\|\omega\|\tau) + \frac{[\omega]_\times^2}{\|\omega\|^2}((1-\cos(\|\omega\|\tau))),$$

Alternatively, we can compute the logarithm which solves for some Lie-algebra element ($\omega \in \mathfrak{so}(3)$ and $\tau \in \mathbb{R}$) given $R \in SO(3)$:

If $\|\omega\| = 1$, then $(\omega, \tau) = \ln R$ is given by:

$$\tau = \cos^{-1}\left(\frac{\mathrm{Tr}(R)-1}{2}\right) \begin{cases} \tau = 0, & \omega \text{ is arbitrary} \\ \tau \neq 0, & \omega = \frac{1}{2\sin(\theta)} \begin{Bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{Bmatrix} \end{cases}$$

If $\|\omega\| \neq 1$, then $\omega = \ln(R)$ is given by:

$$\|\omega\| = \cos^{-1}\left(\frac{\mathrm{Tr}(R)-1}{2}\right)$$
$$\frac{\omega}{\|\omega\|} = \frac{1}{2\sin(\|\omega\|)} \begin{Bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{Bmatrix}$$

We will now extend this to $SE(3)$ and $se(3)$, and then to $SE(2)$ and $se(2)$. This process will look similar except we will obtain:

$$g = e^{\hat{\xi}\tau} \qquad \text{(The exponential map)}$$
$$(\xi, \tau) = \ln g \qquad \text{(The logarithm)}$$

Exponential Representation of Group Motion / Displacements

II. $SE(3)$ and $se(3)$

For this case $\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$ and $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$

We won't derive, just will give the answer:

a) if $\omega = 0$, then

$$\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \implies \hat{\xi}^n = 0 \ \text{ for } n \geq 2$$

Therefore,

$$e^{\hat{\xi}\tau} = \begin{bmatrix} I & v\tau \\ 0 & 1 \end{bmatrix}$$

b) if $\omega \neq 0$, then

$$e^{\hat{\xi}\tau} = \begin{bmatrix} e^{\hat{\omega}\tau} & (I - e^{\hat{\omega}\tau})\frac{\hat{\omega}}{\|\omega\|^2}v + \frac{\omega\omega^\top}{\|\omega\|^2}v\tau \\ 0 & 1 \end{bmatrix}$$

Alternative form for translation:

$$\begin{aligned}
&(I - e^{\hat{\omega}\tau})\frac{\hat{\omega}}{\|\omega\|^2}v + \frac{\omega\omega^\top}{\|\omega\|^2}v\tau \\
&= \left[I - \left(I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\tau) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|\tau))\right)\right]\frac{\hat{\omega}}{\|\omega\|^2}v + \frac{\omega\omega^\top}{\|\omega\|^2}v\tau \\
&= -\left(\frac{\hat{\omega}^2}{\|\omega\|^3}\sin(\|\omega\|\tau) + \frac{\hat{\omega}^3}{\|\omega\|^4}(1 - \cos(\|\omega\|\tau))\right)v + \frac{\omega}{\|\omega\|}\frac{\omega^\top}{\|\omega\|}v\tau \\
&= \frac{\|\omega\|^2\hat{\omega}}{\|\omega\|^4}(1 - \cos(\|\omega\|\tau))v - \frac{\hat{\omega}^2}{\|\omega\|^3}\sin(\|\omega\|\tau)v + \left(I + \frac{\hat{\omega}^2}{\|\omega\|^2}\right)v\tau \\
&= I\tau v + \frac{\hat{\omega}}{\|\omega\|^2}(1 - \cos(\|\omega\|\tau))v + \frac{\hat{\omega}^2}{\|\omega\|^3}(\|\omega\|\tau - \sin(\|\omega\|\tau))v \\
&= \left(I\tau + \frac{\hat{\omega}}{\|\omega\|^2}(1 - \cos(\|\omega\|\tau)) + \frac{\hat{\omega}^2}{\|\omega\|^3}(\|\omega\|\tau - \sin(\|\omega\|\tau))\right)v
\end{aligned}$$

The Logarithm: given $g = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \in SE(3)$, what is $(\xi, \tau) = \ln g$?

Recall that $\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$, so need to figure out for v and for ω.

1. $(\omega, \tau) = \ln R$.

2. use exponent of g to solve for v.

$$\begin{aligned}\vec{d} &= (I-R)\frac{\hat{\omega}}{\|\omega\|^2}v + \frac{\omega\omega^\top}{\|\omega\|^2}v\tau \\ &= \left((I-R)\frac{\hat{\omega}}{\|\omega\|^2} + \frac{\omega\omega^\top}{\|\omega\|^2}\tau\right)v\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}v &= \left((I-R)\frac{\hat{\omega}}{\|\omega\|^2} + \frac{\omega\omega^\top}{\|\omega\|^2}\tau\right)^{-1}\vec{d} \\ &= \|\omega\|^2\left((I-R)\hat{\omega} + \omega\omega^\top\tau\right)^{-1}\vec{d}\end{aligned}$$

or if we use the other method

$$v = \left(I\tau + \frac{\hat{\omega}}{\|\omega\|^2}(1-\cos(\|\omega\|\tau)) + \frac{\hat{\omega}^2}{\|\omega\|^3}(\|\omega\|\tau - \sin(\|\omega\|\tau))\right)^{-1}\vec{d}$$

3. What if $\tau = 0$ or $\omega = 0$? Then no rotation, but there may still be translation.

$$\underbrace{v = d,\ \tau = 1}_{\text{unit time}} \quad \text{or} \quad \underbrace{\tau = \|d\|,\ v = \frac{d}{\|d\|}}_{\text{unit velocity}}$$

Theorem (Chasles)**.** *Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to the axis.*

In other words, the exponential of a twist represents the *relative* motion of a rigid body:

$$\begin{aligned}p(\theta) &= e^{\hat{\xi}\theta}p(0) \\ g_{ab}(\theta) &= e^{\hat{\xi}\theta}g_{ab}(0)\end{aligned}$$

Theorem (Euler)**.** *Any orientation $R \in SO(3)$ is equivalent to a rotation about a fixed axis $\omega \in \mathbb{R}^3$ though an angle $\theta \in [0, 2\pi)$*

III. $SE(2)$ and $se(2)$

exp : for $\xi \in \mathbb{R}^3$ where $\hat{\xi} \in se(2)$,

$$e^{\hat{\xi}\tau} = \begin{cases} \text{if } \omega = 0, & \begin{bmatrix} I & v\tau \\ 0 & 1 \end{bmatrix} \\ \text{if } \omega \neq 0, & \begin{bmatrix} e^{\hat{\omega}\tau} & -\frac{1}{\omega}(I - e^{\hat{\omega}\tau})\mathbb{J}v \\ 0 & 1 \end{bmatrix} \end{cases}$$

where we will use the variable $\mathbb{J}$ to represent the matrix $\mathbb{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

<u>ln:</u> given $g \in SE(2)$,

$$(\omega, \tau) = \ln R$$

if $\omega = 0$ or $\tau = 0$

$$v = \vec{d},\ \tau = 1 \quad \text{or} \quad v = \frac{\vec{d}}{\|d\|},\ \tau = \|d\|$$

else,

$$-\frac{1}{\omega}(I - R)\mathbb{J}v = \vec{d}$$

$\implies$

$$\begin{aligned} v &= -\omega\left[(I - R)\mathbb{J}\right]^{-1}\vec{d} \\ &= -\omega\mathbb{J}^{-1}(I - R)^{-1}\vec{d} \\ &= \omega\mathbb{J}(I - R)^{-1}\vec{d} \end{aligned}$$

Notice that these expressions are the same that are provided in your homework this week:

$$\exp(\hat{\xi}\tau) = \begin{cases} \left[\begin{array}{c|c} R(\xi^3\,\tau) & -\frac{1}{\xi^3}(I - R(\xi^3\tau))\mathbb{J}\begin{Bmatrix}\xi^1 \\ \xi^2\end{Bmatrix} \\ \hline 0 & 1 \end{array}\right] & \text{if } \xi^3 \neq 0 \\ \left[\begin{array}{c|c} I & \begin{Bmatrix}\xi^1 \\ \xi^2\end{Bmatrix}\tau \\ \hline 0 & 1 \end{array}\right] & \text{if } \xi^3 = 0, \end{cases}$$

$$\ln_\tau(g) = \begin{cases} \omega = \frac{1}{\tau}\text{atan}(R_{21}, R_{11}) & \\ v = \omega\mathbb{J}\left(I - R\right)^{-1} d & \text{if } \omega \neq 0\ , \\ v = \frac{1}{\tau}d & \text{if } \omega = 0 \end{cases} \quad \text{with } \xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix},$$

$$\mathbb{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Other Representations of Rotation in $SE(3)$

Exponential coordinates are called the *canonical* coordinates of the rotation group. Other coordinates are Euler angles and Quaternions.

Euler angles

When working in $SO(3)$, there are three elementary rotations about the $x-$, $y-$, and $z-$axes:

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix},$$
$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix},$$
$$R_z(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One common method of utilizing Euler angles to describe the orientation of some coordinate frame B relative to another coordinate frame A is to use the ZYZ convention. This means that the orientation of frame B relative to frame A is described by a triple of angles (α, β, γ) where. These angles describe three rotations: first the rotation of B about the z-axis of B by α, then a rotation about the new y-axis of B by β, and finally a rotation about the new z-axis of B by γ:

$$R_{ba} = R_z(-\gamma)R_y(-\beta)R_z(-\alpha)$$
$$R_{ab} = R_z(\alpha)R_y(\beta)R_z(\gamma)$$

Different ordered sets of rotation axes are also possible. Common choices are ZYX (Fick angles) and YZX axes (Helmholtz angles). The ZYX Euler angles are also referred to as the yaw, pitch, roll angles:

$$R_{ab} = R_z(\psi)R_y(\theta)R_x(\phi).$$

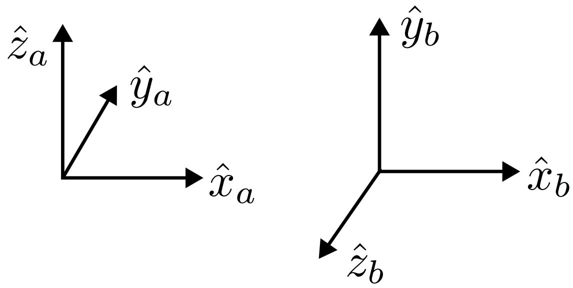
One advantage of these other orders (ZYX and YZX) is that the singularity does *not* occur at the identity orientation ($R = I$). For ZYX Euler angles, the singularity occurs when $\theta = -\frac{\pi}{2}$.

In general, singularities occurs when one degree of freedom is lost and the system becomes "gimbal locked" - essentially, losing the ability to independently control one axis of rotation;

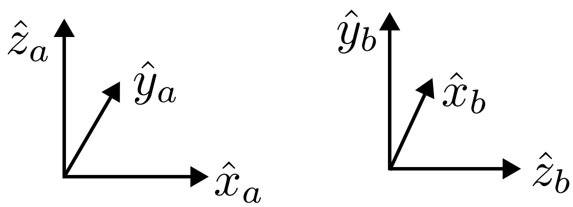
Euler angles are an example of a *local* paramterization of SO(3). It is a fundamental topological fact that singularities can never be eliminated in any 3-dimensional representation of SO(3).

Example

Let's consider the following examples:



$$
\begin{aligned}
R_{ab} &= R_x(\pi/2) \\
&= R_z(0)R_y(0)R_x(\pi/2) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$



$$
\begin{aligned}
R_{ab} &= R_x(\pi/2)Ry(\pi/2) \\
&= R_z(\pi/2)R_y(0)R_x(\pi/2) \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

Quaternions

Unlike Euler angles, quaternions are a *global* representation of SO(3), at the cost of four numbers instead of three. Overall, quaternions are a four-dimensional extension of complex numbers, and can be used to represent rotations in a similar way that complex numbers on the unit circle represent planar rotations. The vector form of a quaternion is:

$$Q = q_0 + q_1 i + q_2 j + q_3 k,$$

where q_0 is the scalar component, and $\vec{q} = (q_1, q_2, q_3)$ is the vector component.

The inverse of a quaternion is given by:

$$
\begin{aligned}
Q^{-1} &= Q^*/\|Q\|^2, \\
Q^* &= (q_0, -\vec{q}), \\
\|Q\|^* &= Q \cdot Q^* = q_0^2 + q_1^2 + q_2^2 + q_3^2.
\end{aligned}
$$

Quaternion products are given by:

$$Q \cdot P = (q_0 p_0 - \vec{q} \cdot \vec{p}, q_0 \vec{p} + p_0 \vec{q} + \vec{q} \times \vec{p}).$$

Unit quaternions are quaternions with a norm of 1 ($\|Q\| = 1$). Notably, unit quaternions form a group with repsect to quaternion multiplication. This allows us to do the following.

Given a rotation matrix $R = \exp(\hat{\omega}\theta)$, the corresponding unit quaternion is:

$$Q = (\cos(\theta/2), \omega \sin(\theta/2))$$

Alternatively, given a unit quaternion $Q = (q_0, \vec{q})$, the corresponding rotation matrix can be computed as:

$$\begin{aligned} R &= \exp(\hat{\omega}\theta), \\ \theta &= 2\cos^{-1}(q_0), \\ \omega &= \begin{cases} \frac{\vec{q}}{\sin(\theta/2)} & \text{if } \theta \neq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For more information on 3D rotations and quaternions, I recommend the following youtube videos:
https://www.youtube.com/watch?v=zjMuIxRvygQ&t=165s
https://www.youtube.com/watch?v=d4EgbgTm0Bg