Topics Covered:

- Review of Rotational Velocity
- Review of Rigid Body Velocity
- Example

Additional Reading:

- MLS Chapter 2, Section 4.2

Review of Angular Velocity

Definition: Angular Velocity

Suppose a frame with unit axes is attached to a rotating body (this frame is called the body frame). The rotation of the frame can be described as a rotation about some unit axis $\hat{\mathrm{w}} \in \mathbb{R}^3$ passing through the origin of the frame.
The rate of rotation of this axis is called the **angular velocity** of the rotating body, defined by:

$$\mathrm{w} = \hat{\mathrm{w}}\dot{\theta},$$

where $\hat{\mathrm{w}}$ is the instantaneous axis of rotation (not yet expressed in any particular reference frame) and $\dot{\theta}$ is the rate of rotation. This equation essentially describes angular velocity in terms of its direction and magnitude.

This angular velocity can be expressed in either the coordinate frame of some other fixed frame (often taken to be the world frame) or the body frame. These definitions are provided next.

Definition: Spatial vs. Body Angular Velocity

Let $R(t)$ be a rotation matrix as seen from some fixed frame, i.e., $R(t) = R_{sb}(t)$, where s is the fixed frame (typically taken to be the world frame) and b is the body frame (attached to the rigid body). Denote $\mathrm{w} \in \mathbb{R}^3$ as the angular velocity of the rotating frame.

Spatial angular velocity is fixed-frame vector representation of w (i.e., the angular velocity expressed in the fixed-frame coordinates which is typically defined to be the world frame) and is defined as:

$$[\omega_s]_\times = \dot{R}R^{-1} = \dot{R}R^\top \in \mathfrak{so}(3)$$

Body angular velocity is body-frame vector representation of w (i.e., the angular velocity expressed in the body frame) and is defined as:

$$[\omega_b]_\times = R^{-1}\dot{R} = R^\top \dot{R} \in \mathfrak{so}(3)$$

It may be helpful to note that the product $\dot{R}_{s\not{b}} R^{-1}_{s\not{b}}$ (for spatial angular velocity) is independent of b and the product $R^{-1}_{\not{s}b}\dot{R}_{\not{s}b}$ (for body angular velocity) is independent of s.

It also might be important to note that ω_b is *not* the angular velocity relative to a moving body frame, but rather it is relative to a *stationary* frame that instantaneously coincides with moving body frame. Otherwise the angular body velocity would always be zero.

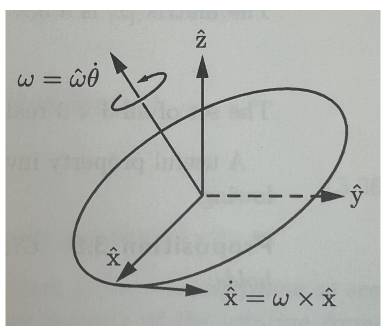
Review of Rigid Body Velocity

In comparison, if I wanted to solve for the *linear velocity* v at any point on a rotating object, I would use the formula:

$$v = \mathrm{w} \times r$$

with r being the position vector from the axis of rotation to the point where I want to solve for velocity.

We can use this formula with our diagram from last week to see how we solved for the rate of rotation for each of our coordinate axes:



where we solved for:

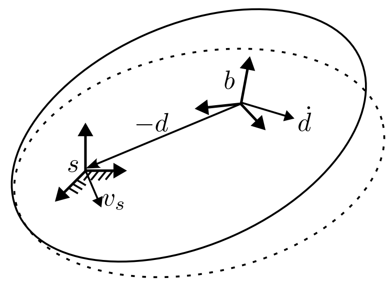
$$\dot{\hat{x}} = \omega \times \hat{x}$$
$$\dot{\hat{y}} = \omega \times \hat{y}$$
$$\dot{\hat{z}} = \omega \times \hat{z}$$

If we take the moving body to be infinitely large, we can interpret the meaning of this by the following diagram and by the following observations:

a) ω_b is the angular velocity expressed in the body frame, and ω_s is the angular velocity expressed in the fixed frame.

b) v_b is the linear vleocity of a point at the origin of b and expressed in b.

c) v_s is the linear velocity of a point at the origin of s and expressed in s.



Definition: Spatial vs Body Linear Velocity

Spatial linear velocity is the instantaneous velocity of the point (if the rigid body was sufficiently large) located at the fixed-frame origin, expressed in the fixed frame. It is defined as:

$$v_s = \dot{d} - \omega_s \times d = \dot{d} - \dot{R}R^\top d$$

Note: v_s is *not* the linear velocity of the body-fixed frame origin expressed in the fixed frame that quantity would simply be $\dot{d}$.

Body linear velocity is the velocity of a rigid body with respect to the (instantaneous) body frame. It is defined as:

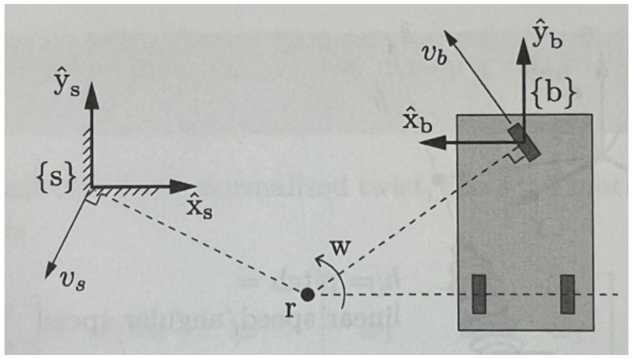
$$v_b = R_{sb}^{-1}\dot{d} = R^\top \dot{d}$$

Derivations for these terms can be obtained through our derivation of the spatial and body twists (note, this was done in previous lectures):

$$\hat{\xi}_s = \dot{g}_{sb} g_{sb}^{-1}$$
$$\hat{\xi}_b = g_{sb}^{-1}\dot{g}_{sb}$$

Example Revisited

Consider the following example:



This example shows the top view of a car, with a single steerable front wheel driving on a plane. The angular velocity caused the the rotation of the front wheel is $\mathbf{w} = 2$ rad/s about point r. We can write the point of rotation in reference to either s or b as:

$$r_s = (2, -1, 0), \quad r_b = (2, -1.4, 0)$$

The angular velocity can then be expressed in either frame as:

$$\omega_s = (0, 0, 2), \quad \omega_b = (0, 0, -2)$$

From the figure, we can solve for spatial and body velocity of the car's frame as:

$$v_s = \omega_s \times (-r_s) = r_s \times \omega_s = (-2, -4, 0) = \begin{Bmatrix} -2 \\ -4 \\ 0 \\ 0 \\ 0 \\ 2 \end{Bmatrix}$$

$$v_b = \omega_b \times (-r_b) = r_b \times \omega_b = (2.8, 4, 0) = \begin{Bmatrix} 2.8 \\ 4 \\ 0 \\ 0 \\ 0 \\ -2 \end{Bmatrix}$$

Putting these together we can obtain the spatial and body twists as:

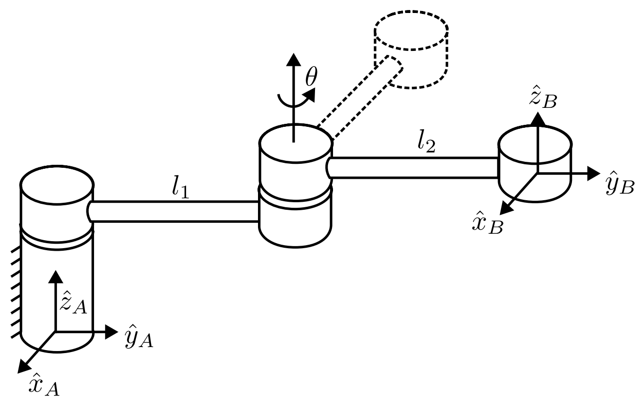
$$\xi_s = \begin{Bmatrix} -2 \\ -4 \\ 2 \end{Bmatrix}, \quad \xi_b = \begin{Bmatrix} 2.8 \\ 4 \\ -2 \end{Bmatrix}$$

Notice that we can also do this in 2D notation using $[\omega]_\times = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$:

$$v_s = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$
$$v_b = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1.4 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 4 \end{bmatrix}$$

Example

(Example 2.5 from MLS Chapter 2.4)



We can derive the configuration of this robot as:

$$\begin{aligned}
g(t) &= g_{at} g_{tb} \qquad \text{(Note: } t \text{ represents the temporary frame at the joint location)} \\
&= \begin{bmatrix} R(\theta(t)) & \begin{bmatrix} 0 \\ l_1 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(0) & \begin{bmatrix} 0 \\ l_2 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R(\theta(t))R(0) & \begin{bmatrix} 0 \\ l_1 \end{bmatrix} + R(\theta(t)) \begin{bmatrix} 0 \\ l_2 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R(\theta(t)) & \begin{bmatrix} 0 \\ l_1 \end{bmatrix} + \begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix} \begin{bmatrix} 0 \\ l_2 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) & -l_2 \sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) & l_1 + l_2 \cos(\theta(t)) \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

The spatial velocity of the rotating rigid body can be solved for as:

$$\xi_s = \begin{Bmatrix} v_s \\ \omega_s \end{Bmatrix} = \begin{Bmatrix} -\dot{R}R^\top p + \dot{p} \\ (\dot{R}R^\top)^\vee \end{Bmatrix}$$

where p denotes the position of the body frame origin in the fixed frame:

$$p(t) = \begin{bmatrix} -l_2 \sin(\theta(t)) \\ l_1 + l_2 \cos(\theta(t)) \end{bmatrix}$$

and R is the orientation of the body frame origin in the fixed frame:

$$R(t) = \begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix}$$

If we solve for these expressions you should get:

$$v_s = \begin{Bmatrix} l_1\dot{\theta} \\ 0 \end{Bmatrix}, \quad \omega_s = \dot{\theta}$$

This computation was derived in a previous class using the homogeneous representation:

$$\begin{aligned} \hat{\xi}_s &= \dot{g}g^{-1} \\ &= \begin{bmatrix} \dot{R}R^\top & -\dot{R}R^\top p + \dot{p} \\ 0 & 0 \end{bmatrix} \qquad \text{(Note that the form of this matrix is } \begin{bmatrix} [\omega_s]_\times & v_s \\ 0 & 0 \end{bmatrix}\text{)} \\ &= \begin{bmatrix} 0 & -\dot{\theta} & l_1\dot{\theta} \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

We can see how we got this by solving for the terms:

$$\begin{aligned} \dot{R}R^\top &= \begin{bmatrix} -\sin(\theta(t))\dot{\theta} & -\cos(\theta(t))\dot{\theta} \\ \cos(\theta(t))\dot{\theta} & -\sin(\theta(t))\dot{\theta} \end{bmatrix} \begin{bmatrix} \cos(\theta(t)) & \sin(\theta(t)) \\ -\sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \\ -\dot{R}R^\top p + \dot{p} &= \begin{bmatrix} 0 & \dot{\theta} \\ -\dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} -l_2\sin(\theta(t)) \\ l_1 + l_2\cos(\theta(t)) \end{bmatrix} + \begin{bmatrix} -l_2\cos(\theta(t))\dot{\theta} \\ -l_2\sin(\theta(t))\dot{\theta} \end{bmatrix} \\ &= \begin{bmatrix} l_1\dot{\theta} + \cancel{l_2\cos(\theta(t))\dot{\theta}} - \cancel{l_2\cos(\theta(t))(\dot{\theta})} \\ \cancel{l_2\sin(\theta(t))\dot{\theta}} - \cancel{l_2\sin(\theta(t))\dot{\theta}} \end{bmatrix} = \begin{bmatrix} l_1\dot{\theta} \\ 0 \end{bmatrix} \end{aligned}$$

Again, note that the interpretation behind spatial velocity is that v_s is the velocity of a point attached to the rigid body as it travels through the origin of the A coordinate frame.

To solve for the body velocity, we follow a similar procedure:

$$\xi_b = \begin{Bmatrix} v_b \\ \omega_b \end{Bmatrix} = \begin{Bmatrix} R^\top \dot{p} \\ (R^\top \dot{R})^\vee \end{Bmatrix}$$

If we solve for these expressions you should get:

$$v_b = \begin{Bmatrix} -l_2\dot{\theta} \\ 0 \end{Bmatrix}, \quad \omega_s = \dot{\theta}$$

As before, this computation can also be derived using homogeneous representation:

$$\begin{aligned}\hat{\xi}_b &= g^{-1}\dot{g} \\ &= \begin{bmatrix} R^\top \dot{R} & R^\top \dot{p} \\ 0 & 0 \end{bmatrix} \qquad \text{(Note that the form of this matrix is } \begin{bmatrix} [\omega_b]_\times & v_b \\ 0 & 0 \end{bmatrix}\text{)} \\ &= \begin{bmatrix} 0 & -\dot{\theta} & -l_2\dot{\theta} \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

Again, we can check our computations:

$$\begin{aligned} R^\top \dot{R} &= \begin{bmatrix} \cos(\theta(t)) & \sin(\theta(t)) \\ -\sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix} \begin{bmatrix} -\sin(\theta(t))\dot{\theta} & -\cos(\theta(t))\dot{\theta} \\ \cos(\theta(t))\dot{\theta} & -\sin(\theta(t))\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \\ R^\top \dot{p} &= \begin{bmatrix} \cos(\theta(t)) & \sin(\theta(t)) \\ -\sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix} \begin{bmatrix} -l_2\cos(\theta(t))\dot{\theta} \\ -l_2\sin(\theta(t))\dot{\theta} \end{bmatrix} = \begin{bmatrix} -l_2\dot{\theta} \\ 0 \end{bmatrix} \end{aligned}$$

The body velocity should be interpreted by imagining the velocity of the origin of the B frame, as seen by the B coordinates, so linear velocity is always in the $-x$ direction and angular velocity is always in the z direction (for this problem!).

Change of Reference Frame

We can change our reference frame from body to spatial by using adjoint transformation:

$$\xi_s = \mathrm{Ad}_g \xi_b \qquad \text{(Note here that } g \text{ is } g_{sb}\text{)}$$

Where this adjoint is performed the same way as we learned earlier in the course:

$$\begin{aligned}\hat{\xi}_s &= g_{sb}\hat{\xi}_b g_{sb}^{-1} \\ &= \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\omega_b]_\times & v_b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^\top & -R^\top d \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^\top \dot{R} & R^\top \dot{d} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^\top & -R^\top d \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} RR^\top \dot{R} & RR^\top \dot{d} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^\top & -R^\top d \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{R}R^\top & -\dot{R}R^\top d + \dot{d} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} [\omega_s]_\times & v_s \\ 0 & 0 \end{bmatrix}\end{aligned}$$