**Topics Covered:**

- Subscript Cancellation Rule
- Angular Velocity
- Twists
- Example

**Additional Reading:**

- LP 3.2.2 (Angular Velocity), 3.3.2 (Twists)
- MLS Chapter 2, Section 4

**Definition: Subscript Cancellation Rule**

Page 62 of Modern Robotics by Kevin Lynch and Frank Park (LP) describes the "Subscript Cancellation Rule" as follows: When multiplying two rotation matrices, if the second subscript of the first matrix matches the first subscript of the second matrix, the two subscripts "cancel" and a change of reference frame is achieved:

$$R_{ab}R_{bc} = R_{a\cancel{b}}R_{\cancel{b}c} = R_{ac}$$

A rotation matrix is just a collection of three unit vectors, so the reference frame of a vector can also be changed by a rotation matrix using a modified version of the Subscript Cancellation Rule:

$$R_{ab}p_b = R_{a\cancel{b}}p_{\cancel{b}} = p_a$$

The subscript cancellation rule also extends to transformations by considering some arbitrary reference frames $a$, $b$ and $c$ and some vector $v$ expressed in frame $b$:

$$g_{ab}g_{bc} = g_{a\cancel{b}}g_{\cancel{b}c} = g_{ac}$$

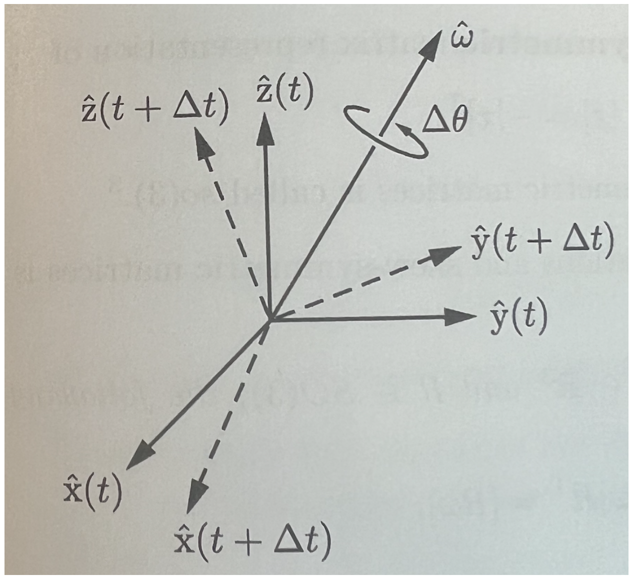
and

$$g_{ab}v_b = g_{a\cancel{b}}v_{\cancel{b}} = v_a$$

where $v_a$ is the vector $v$ expressed in frame $a$.

# Angular Velocity

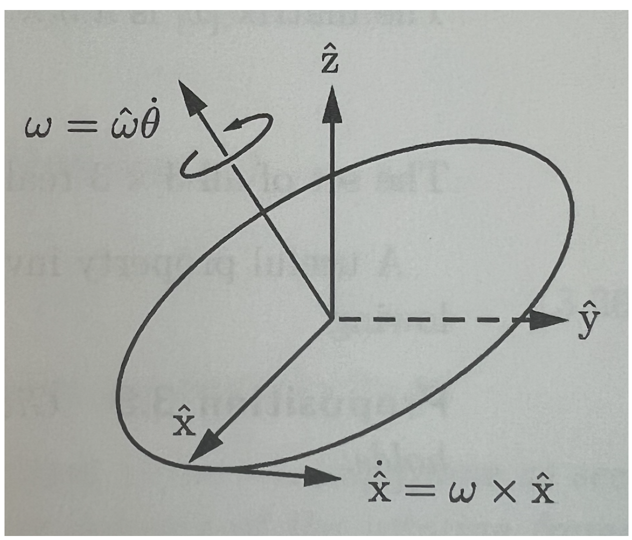
Consider the following frame attached to a rotating body:




If we examine the body frame at times $t$ and $t+\delta t$, the change in frame orientation can be described as a rotation of angle $\delta\theta$ about some unit axis $\hat{\omega}$ passing through the origin. As the limit $\delta t \to 0$, we can define the angular velocity $\omega$ as:

$$\omega = \hat{w}\dot{\theta}$$

This angular velocity is illustrated as follows:




From the figure, we can decompose the individual coordinate axis velocities as:

$$\dot{\hat{x}} = \omega \times \hat{x}$$
$$\dot{\hat{y}} = \omega \times \hat{y}$$
$$\dot{\hat{z}} = \omega \times \hat{z}$$

To express these equations in coordinates, we must choose a reference frame for $\omega$. Two natural choices are the fixed frame $s$ and the body frame $b$. This will later be discussed as resulting in either *spacial velocity* or *body velocity*.

Starting with spacial frame, let $\omega_s \in \mathbb{R}^3$ be the angular velocity expressed in fixed-frame coordinates. Additionally let $R(t)$ be the rotation matrix describing the orientation of the body frame with respect to the fixed frame at time $t$ (i.e., $R_{sb}(\theta(t))$). Each column of $R$ then denotes a coordinate frame axis in fixed-frame coordinates, denotes as

$$R = \begin{bmatrix} | & | & | \\ r_1 & r_2 & r_3 \\ | & | & | \end{bmatrix}$$

Thus, the time rate of change for $R$ can be expressed as:

$$\dot{R} = \begin{bmatrix} \omega_s \times r_1 & \omega_s \times r_2 & \omega_s \times r_3 \end{bmatrix} = \omega_s \times R$$

**Aside into cross products:**

Given two vectors $a$ and $b \in \mathbb{R}^3$, the cross product $a \times b$ represents the vector that is orthogonal (perpendicular) to both $a$ and $b$ with the direction determined by the right-hand rule. This product is defined as the determinant of:

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where $i$, $j$, and $k$ are unit vectors in $x$, $y$, and $z$ directions, respectively.
This is equivalent to the expression:

$$a \times b = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

We're going to use our trick with skew-symmetric matrices to get rid of the cross product.

**Skew symmetric matrices:**

Given a vector $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top \in \mathbb{R}^3$, then the skew-symmetric matrix for $x$ is

$$[x]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Thus, by converting $\omega_s$ into a skew-symmetric matrix, we can eliminate the cross product as:

$$\begin{aligned} \dot{R} &= \omega_s \times R \\ &= [\omega_s]_\times R \end{aligned}$$

**Skew symmetric matrices for planar rotations:**

Note, in 2D our element $\omega$ is one-dimensional, so our "cross product" is equivalent to rotating a vector in the plane. This rotation can be conceptualized as a *perpendicular operator* represented by the skew-symmetric matrix:

$$[\omega]_\times = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

Thus, for 2D, we denote $\omega \times R$ as:

$$\omega \times R = [\omega]_\times R$$

Continuing with our previous expression for the time rate of change of $R$, we can express the angular velocity in fixed-frame coordinates by post-multiplying both sides by $R^{-1}$ gives us the expresson for the spacial angular velocity:

$$\begin{aligned} \dot{R} &= \omega_s \times R \\ &= [\omega_s]_\times R \\ [\omega_s]_\times &= \dot{R}R^{-1} \end{aligned}$$

Now, if we consider $\omega_b$ to be expressed in body-frame coordinates, we can solve for the body frame velocity using the transformation:

$$\omega_s = R_{sb}\omega_b$$

Rearranging this expression also gives us the expression for the body angular velocity:

$$\omega_b = R_{sb}^{-1}\omega_s = R^{-1}\omega_s = R^\top \omega_s$$

Expressing the body-frame angular velocity in skew-symmetric matrix form yields:

$$\begin{aligned} [\omega_b]_\times &= [R^\top \omega_s]_\times \\ &= R^\top [\omega_s]_\times R && \text{(proof from Prop 3.8 of LP)} \\ &= R^\top (\dot{R}R^\top) R \\ &= R^\top \dot{R} = R^{-1}\dot{R} \end{aligned}$$

**Property of skew-symmetric matrices:**

(From Proposition 3.8 of LP)
Given any $\omega \in \mathbb{R}^3$, and $R \in SO(3)$, the following always holds:

$$R^\top [\omega]_\times R = [R^\top \omega]_\times$$

*Proof.* Letting $r_i^\top$ be the $i$th row of $R$, we have:

$$
\begin{aligned}
R^\top [\omega]_\times R &= \begin{bmatrix} r_1^\top \\ r_2^\top \\ r_3^\top \end{bmatrix} \begin{bmatrix} \omega \times r_1 & \omega \times r_2 & \omega \times r_3 \end{bmatrix} \\
&= \begin{bmatrix} r_1^\top(\omega \times r_1) & r_1^\top(\omega \times r_2) & r_1^\top(\omega \times r_3) \\ r_2^\top(\omega \times r_1) & r_2^\top(\omega \times r_2) & r_2^\top(\omega \times r_3) \\ r_3^\top(\omega \times r_1) & r_3^\top(\omega \times r_2) & r_3^\top(\omega \times r_3) \end{bmatrix} \\
&= \begin{bmatrix} 0 & -r_3^\top \omega & r_2^\top \omega \\ r_3^\top \omega & 0 & -r_1^\top \omega \\ -r_2^\top \omega & r_1^\top \omega & 0 \end{bmatrix} \\
&= [R^\top \omega]_\times
\end{aligned}
$$

**Definition: Angular Velocity in Fixed-Frame and Body-Frame (Prop 3.9 of LP)**

Let $R(t)$ denote the orientation of the rotating frame as seen from the fixed frame. Denote $\omega$ as the angular velocity of the rotating frame. Then:

$$
\begin{aligned}
[\omega_s]_\times &= \dot{R} R^{-1} \\
[\omega_b]_\times &= R^{-1} \dot{R} = R^\top \dot{R}
\end{aligned}
$$

# Twists

Inspired by the definition of $[\omega_b]_\times$, let's apply this formula to our homogeneous representation of transformations:

$$
\begin{aligned}
g^{-1} \dot{g} &= \begin{bmatrix} R^\top & -R^\top d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{d} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} R^\top \dot{R} & R^\top \dot{d} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} [\omega_b]_\times & v_b \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

We will call this vector as the **body twist**, defined as:

$$
\xi = \begin{Bmatrix} v_b \\ \omega_b \end{Bmatrix}, \quad \hat{\xi} = \begin{bmatrix} [\omega_b]_\times & v_b \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)/\mathfrak{se}(3)
$$

If we instead, consider the angular velocity in fixed-frame coordinates, we can derive the **spatial**

**twist** as:

$$\begin{aligned}\dot{g}g^{-1} &= \begin{bmatrix}\dot{R} & \dot{d}\\ 0 & 0\end{bmatrix}\begin{bmatrix}R^\top & -R^\top d\\ 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}\dot{R}R^\top & -\dot{R}R^\top d+\dot{d}\\ 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}[\omega_s]_\times & v_s\\ 0 & 0\end{bmatrix}\end{aligned}$$

The intuition behind the expression for $v_s$ can be derived as:

$$\begin{aligned}v_s &= \dot{d}-\dot{R}R^\top d\\ &= \dot{d}-[\omega_s]_\times d\\ &= \dot{d}-\omega_s\times d\\ &= \dot{d}+\omega_s\times -d\end{aligned}$$

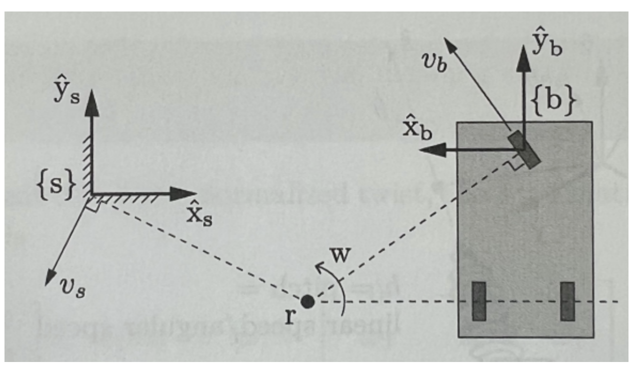
In words, this is the "instantaneous velocity of the point on the body currently at the fixed-frame origin, expressed in the fixed frame".

So, the **spatial twist** is defined as:

$$\xi = \begin{Bmatrix}v_s\\ \omega_s\end{Bmatrix}, \quad \hat{\xi} = \begin{bmatrix}[\omega_s]_\times & v_s\\ 0 & 0\end{bmatrix} \in \mathfrak{se}(2)/\mathfrak{se}(3)$$

# Example

Consider the following example:




This example shows the top view of a car, with a single steerable front wheel driving on a plane. The angular velocity caused the the rotation of the front wheel is $\mathbf{w} = 2$ rad/s about point $r$. We can write the point of rotation in reference to either $s$ or $b$ as:

$$r_s = (2,-1,0), \quad r_b = (2,-1.4,0)$$

The angular velocity can then be expressed in either frame as:

$$\omega_s = (0,0,2), \quad \omega_b = (0,0,-2)$$

From the figure, we can solve for spatial and body velocity of the car's frame as:

$$v_s = \omega_s \times (-r_s) = r_s \times \omega_s = (-2,-4,0) = \begin{Bmatrix} -2 \\ -4 \\ 0 \\ 0 \\ 0 \\ 2 \end{Bmatrix}$$

$$v_b = \omega_b \times (-r_b) = r_b \times \omega_b = (2.8,4,0) = \begin{Bmatrix} 2.8 \\ 4 \\ 0 \\ 0 \\ 0 \\ -2 \end{Bmatrix}$$

Putting these together we can obtain the spatial and body twists as:

$$\xi_s = \begin{Bmatrix} -2 \\ -4 \\ 2 \end{Bmatrix}, \quad \xi_b = \begin{Bmatrix} 2.8 \\ 4 \\ -2 \end{Bmatrix}$$

Notice that we can also do this in 2D notation using $[\omega]_\times = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$:

$$v_s = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$
$$v_b = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1.4 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 4 \end{bmatrix}$$

Next class we will go over how to apply change of frame transformations to twists using the adjoint operation, i.e., $\hat{\xi}_s = \mathrm{Ad}_{g_{sb}} \hat{\xi}_b$.