Topics Covered:

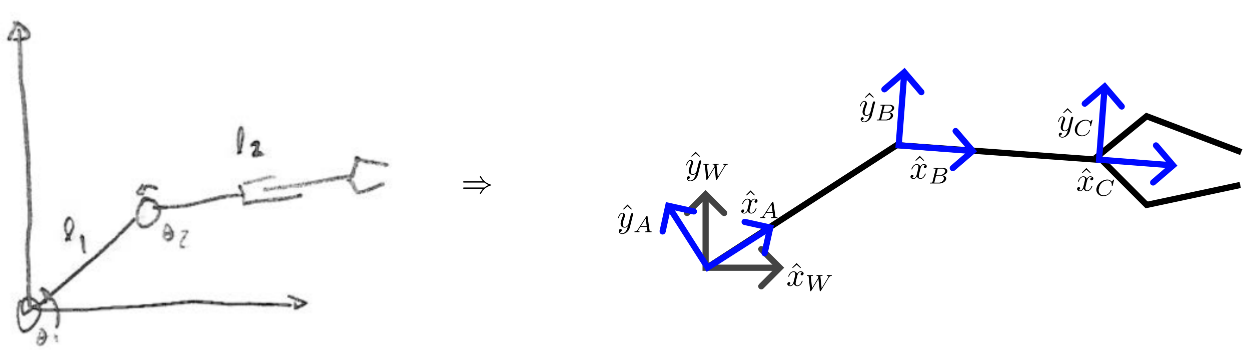
- Special Euclidean Group $SE(2)$
- Homogeneous Coordinates
- Planar Manipulator Example

Additional Reading:

- LP 3.2.1, 3.3.1
- MLS 3.1

Review

Last class we introduced the product structure of transformations. We applied this concept to the following example:



where we solved for the end-effector configuration g_{WC} (the transformation of frame C with resepct to the world frame) as:

$$\begin{aligned} g_{WC} &= g_{WA} \cdot g_{AB} \cdot g_{BC} \\ &= (0, R(\theta_1)) \cdot (\vec{d}_1, R(\theta_2)) \cdot (\vec{d}_2, I) \\ &= (0, R(\theta_1)) \cdot (\vec{d}_1 + R(\theta_2)\vec{d}_2, R(\theta_2)) \\ &= (R(\theta_1)\vec{d}_1 + R(\theta_1)R(\theta_2)\vec{d}_2, R(\theta_1)R(\theta_2)) \end{aligned}$$

Today we will review the Special Euclidean Group $SE(2)$ and introduce homogeneous coordinates. These homogeneous coordinates will simplify our calculations.

Special Euclidean Group $SE(2)$

The space of planar rigid body configurations / transformations is called $SE(2)$ (termed Special Euclidean).

Special Euclidean Group $SE(2)$:

1. closure	$g_1 \cdot g_2 \in G$
2. associativity	$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3))$
3. identity element exists	$e = (0, I)$
4. inverse exists	$g^{-1} = (-R^T \vec{d}, R^T)$

↳ note that we saw multiple representations for $SE(2)$, want to consider a special version, called homogeneous representation. This will make our computations more convenient.

It is perhaps also interesting to note that the group $SE(2)$ is an instance of a Lie Group.

Definition: Lie Group

A Lie group is a group G which is also a smooth manifold and for which the group product and inverse are smooth.

Homogeneous Coordinates

Homogeneous coordinates translate a transformation into a matrix form:

$$(\vec{d}, R) \to \left[\begin{array}{c|c} R & \vec{d} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \times & \times & \times \\ \times & \times & \times \\ \hline \times & \times & 1 \end{array}\right] \quad \text{matrix}$$

where the matrix on the right illustrates the sizes of each element:

$$R \to 2 \times 2, \quad \vec{d} \to 2 \times 1, \quad 0 \to 1 \times 2, \quad 1 \to 1 \times 1$$

We can demonstrate that the properties of the $SE(2)$ group still are valid for homogeneous coordinates.

Properties of the $SE(2)$ group:

Closure

$$g_1 g_2 = \begin{bmatrix} R_1 & \vec{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & \vec{d}_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 \vec{d}_2 + \vec{d}_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & \vec{d}_1 + R_1 \vec{d}_2 \\ 0 & 1 \end{bmatrix}$$

$$\text{vs. } (\vec{d}_1, R_1) \cdot (\vec{d}_2, R_2) = (\vec{d}_1 + R_1\vec{d}_2, R_1R_2)$$

Associativity Matrix multiplication preserves associativity $(AB)C = A(BC)$.
Identity element:

$$e = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse element: We will skip the proof of the matrix inversion for now, but the computation would arrive at the following form for the inverse element:

$$g^{-1} = \begin{bmatrix} R^T & -R^T\vec{d} \\ 0 & 1 \end{bmatrix}$$

Applying homogeneous coordinates to points

What about how we apply transformations to points and vectors?

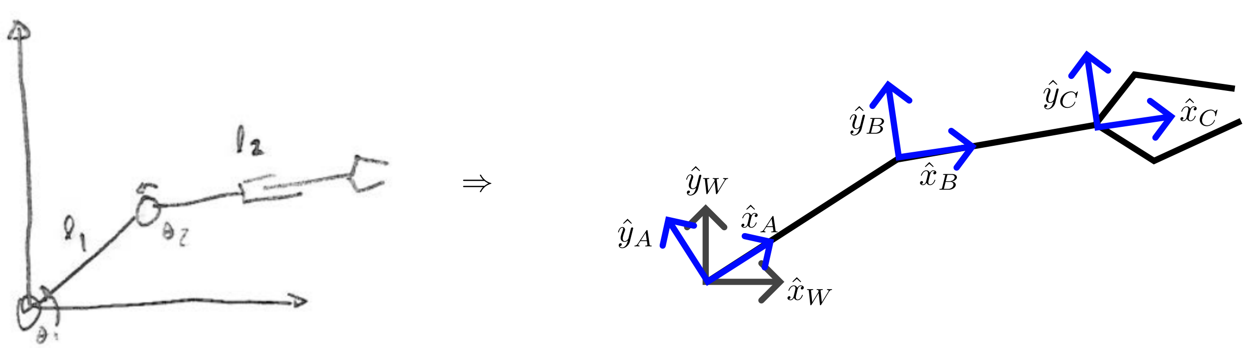
We will now represent points by $\begin{Bmatrix} p \\ 1 \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}$

To transform a point using homogeneous coordinates, we perform matrix multiplication:

$$g \cdot p = \begin{bmatrix} R & \vec{d} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} p \\ 1 \end{Bmatrix} = \begin{Bmatrix} Rp + \vec{d} \\ 1 \end{Bmatrix}$$

Manipulators and SE(2)

Consider the same example planar manipulator as before:



We can solve for the end-effector configuration g_{WC} using the homogeneous coordinates:

$$\begin{aligned} g_{WC} = g_{WA} \cdot g_{AB} \cdot g_{BC} &= \begin{bmatrix} R(\theta_1) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta_2) & \vec{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \vec{d}_2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R(\theta_1)R(\theta_2) & R(\theta_1)\vec{d}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \vec{d}_2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R(\theta_1)R(\theta_2) & R(\theta_1)R(\theta_2)\vec{d}_2 + R(\theta_1)\vec{d}_1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Observe that this matches the previous coordinates we obtained:

$$\text{i.e., } g_{WC} = (R(\theta_1)\vec{d}_1 + R(\theta_1)R(\theta_2)\vec{d}_2, R(\theta_1)R(\theta_2))$$

Note: we can make this computation slightly easier by observing that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$. Be careful, this is only true when the rotation axes are the same. It comes from the fact that $\cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$ and $\sin(\theta_1+\theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)$.

Proof that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$:

$$\begin{aligned} R(\theta_1)R(\theta_2) &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \\ &= R(\theta_1+\theta_2) \end{aligned}$$

Thus, we can equivalently write our end-effector configuration as:

$$g_{WC} = \begin{bmatrix} R(\theta_1+\theta_2) & R(\theta_1+\theta_2)\vec{d}_2 + R(\theta_1)\vec{d}_1 \\ 0 & 1 \end{bmatrix}$$

Example Let's now consider the planar manipulator with specific parameters. Assume that the manipulator is designed such that $l_1 = 1$, $l_2 \in [\frac{1}{2}, 2]$, $\theta_1 \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, $\theta_2 \in [\frac{-3\pi}{4}, \frac{3\pi}{4}]$

In the zero configuration, our displacement variables are defined as:

$$\vec{d}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \vec{d}_2 = \begin{Bmatrix} l_2 \\ 0 \end{Bmatrix}$$

Question: What is the end effector configuration for: $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{\pi}{6}$, $l_2 = 1$?

$$
\begin{aligned}
g_{WC} &= \begin{bmatrix} R(\theta_1+\theta_2) & R(\theta_1+\theta_2)\vec{d}_2 + R(\theta_1)\vec{d}_1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R(\frac{\pi}{3}) & R(\frac{\pi}{6})\begin{Bmatrix}1\\0\end{Bmatrix} + R(\frac{\pi}{3})\begin{Bmatrix}1\\0\end{Bmatrix} \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Solving for the displacement term gives us:

$$
\begin{aligned}
R\left(\frac{\pi}{6}\right)\begin{Bmatrix}1\\0\end{Bmatrix} + R\left(\frac{\pi}{3}\right)\begin{Bmatrix}1\\0\end{Bmatrix} &= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix} + \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix} \\
&= \begin{Bmatrix}(1+\sqrt{3})/2 \\ (1+\sqrt{3})/2\end{Bmatrix}
\end{aligned}
$$

So plugging in this simplified expression gives us:

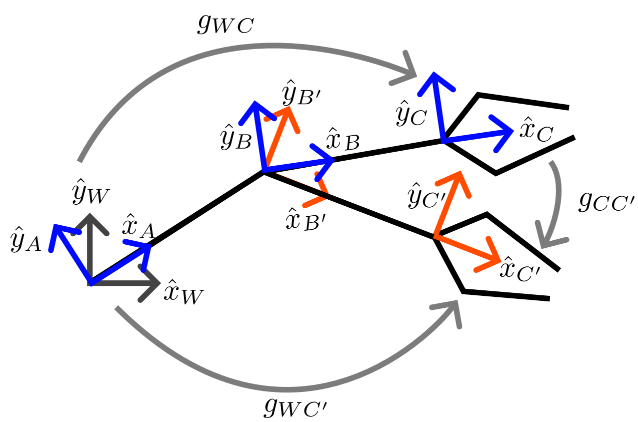
$$
g_{WC} = \begin{bmatrix} R(\pi/3) & \begin{Bmatrix}(1+\sqrt{3})/2 \\ (1+\sqrt{3})/2\end{Bmatrix} \\ 0 & 1 \end{bmatrix} \equiv (\underbrace{(1+\sqrt{3})/2}_{x}, \underbrace{(1+\sqrt{3})/2}_{y}, \underbrace{\pi/3}_{\theta})
$$

Question: If the end-effector then grabs something and moves to $\theta_1 = \frac{\pi}{6}$, $\theta_2 = -\frac{\pi}{6}$, $l_2 = 2$. What is the end-effector configuration now?

$$
\begin{aligned}
g_{WC} &= \begin{bmatrix} R(0) & R(\pi/6)\begin{Bmatrix}1\\0\end{Bmatrix} + R(0)\begin{Bmatrix}2\\0\end{Bmatrix} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} & \begin{bmatrix}\sqrt{3}/2 & -0.5 \\ 0.5 & \sqrt{3}/2\end{bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix} + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}2\\0\end{Bmatrix} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} I & \begin{Bmatrix}2+\sqrt{3}/2 \\ 0.5\end{Bmatrix} \\ 0 & 1 \end{bmatrix} \equiv \left(\frac{4+\sqrt{3}}{2}, 0.5, 0\right)
\end{aligned}
$$

Question: What transformation did the end effector undergo?

Well, we can consider the transformation pictorially as:



where g_{WC} represents the first configuration we solved for in reference to the world frame W, and $g_{WC'}$ represents the second configuration, again in reference to the world frame W.

Following the arrows, we see that we can solve for the transformation that the end-effector undergoes as:

$$g_{CC'} = (g_{WC})^{-1} g_{WC'}$$

Plugging in our homogeneous coordinate for g_{WC} and $g_{WC'}$ yields:

$$\begin{aligned}
g_{CC'} &= (g_{WC})^{-1} g_{WC'} \\
&= \begin{bmatrix} R(-\frac{\pi}{3}) & -R(-\frac{\pi}{3}) \begin{Bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(0) & \begin{Bmatrix} 2+\frac{\sqrt{3}}{2} \\ 1/2 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \qquad \text{(Using: } g^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}\text{)} \\
&= \begin{bmatrix} R(-\frac{\pi}{3}) & R(-\frac{\pi}{3}) \begin{Bmatrix} 2+\frac{\sqrt{3}}{2} \\ 1/2 \end{Bmatrix} - R(-\frac{\pi}{3}) \begin{Bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R(-\frac{\pi}{3}) & R(-\frac{\pi}{3}) \begin{Bmatrix} 3/2 \\ -\frac{\sqrt{3}}{2} \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R(-\frac{\pi}{3}) & \begin{Bmatrix} 0 \\ -2\sqrt{3} \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \equiv (0, -2\sqrt{3}, -\frac{\pi}{3})
\end{aligned}$$

Question: Lastly, now consider that there's an object in the end-effector's grip. What would this transformation ($g_{CC'}$) be for the object?

We can solve for this transformation using the adjoint transformation, which was defined in a previous lecture as:

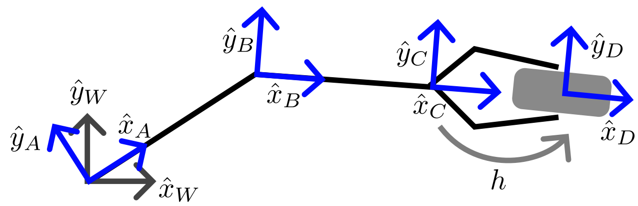
$$\text{Ad}_h g = h g h^{-1}$$

This is also known as the adjoint transformation associated with h applied to g.

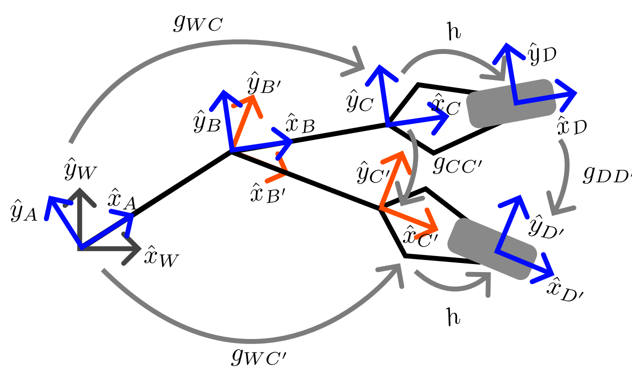
In our case, consider the transformation from the end-effector to the object as:

$$h = \begin{bmatrix} I & \begin{Bmatrix} 1/4 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

which is illustrated by the following diagram:



We can solve for our formulation of the adjoint by following the arrows in the following diagram:



which gives us the relationship:

$$g_{DD'} = h^{-1} g_{CC'} h = \mathrm{Ad}_{h^{-1}} g_{CC'}$$

Solving for this expression yields:

$$\begin{aligned} g_{DD'} &= \begin{bmatrix} I & d_h \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} R(-\pi/3) & \begin{Bmatrix} 0 \\ -2\sqrt{3} \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & d_h \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} I & -d_h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(-\pi/3) & \begin{Bmatrix} 0 \\ -2\sqrt{3} \end{Bmatrix} + R(-\pi/3) d_h \\ 0 & 1 \end{bmatrix} \qquad \text{(Using: } g^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}\text{)} \end{aligned}$$

Solving separately for $R(-\pi/3)d_h$ yields:

$$\begin{aligned} R(-\pi/3)d_h &= \begin{bmatrix} \cos(-\pi/3) & -\sin(-\pi/3) \\ \sin(-\pi/3) & \cos(-\pi/3) \end{bmatrix} \begin{Bmatrix} 1/4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1/4 \\ -\sqrt{3}/4 \end{Bmatrix} \\ &= \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \begin{Bmatrix} 1/4 \\ 0 \end{Bmatrix} \\ &= \begin{Bmatrix} 1/8 \\ -\sqrt{3}/8 \end{Bmatrix} \end{aligned}$$

Plugging this back in:

$$\begin{aligned} g_{DD'} &= \begin{bmatrix} I & -d_h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(-\pi/3) & \begin{Bmatrix} 0 \\ -2\sqrt{3} \end{Bmatrix} + \begin{Bmatrix} 1/8 \\ -\sqrt{3}/8 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} I & -d_h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(-\pi/3) & \begin{Bmatrix} 1/8 \\ -(2+1/8)\sqrt{3} \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R(-\pi/3) & \begin{Bmatrix} 1/8 \\ -(2+1/8)\sqrt{3} \end{Bmatrix} - \begin{Bmatrix} 1/4 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R(-\pi/3) & \begin{Bmatrix} -1/8 \\ -(2+1/8)\sqrt{3} \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Thus, the object in the end-effector's grip undergoes the transformation:

$$(-1/8, -(2+1/8)\sqrt{3}, -\pi/3)$$

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