Topics Covered:

- Product Structure of Transformations
- Inverse Transformation

Additional Reading:

- Lynch, K.M. and Park, F.C. Modern Robotics: Section 3.3.1
- Craig, J.J. Introduction to Robotics: 2.3
- Murray et al. A Mathematical Introduction to Robotic Manipulation: Chapter 2, Section 3.1

Notation

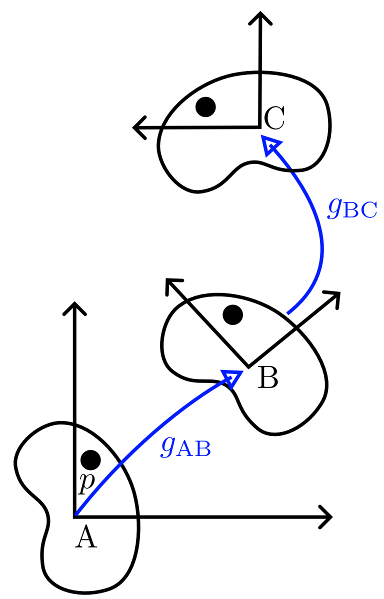
For the sake of consistent notation moving forward, we will define and use the following:

- $\hat{x}_A$: the unit vector in the x direction of frame A
- $\hat{y}_A$: the unit vector in the y direction of frame A
- g_{AB}: the transformation from frame A to frame B
- $\vec{d}^{A}_{AB}$: the displacement from frame A to frame B in reference to frame A
- $R(\theta_{AB})$: the rotation matrix that rotates a point by θ_{AB} radians, with θ_{AB} being the angle from frame A to frame B
- $\vec{p}^{\mathrm{A}}_{\mathrm{B}}$: a point p in frame B that is represented in frame A
- $\vec{v}$: general vector notation for some variable v

Review

Last class we (tried) to introduce how you can conduct multiple transformations. To review, let's consider the following example:

Multiple Displacements Derivation



when a rigid body experiences a displacement g, the point $\vec{p}$ undergoes a transformation:

$$\vec{p}\,' = g \cdot \vec{p} = (\vec{d}, R) \cdot \vec{p} = \underbrace{\vec{d}}_{\text{translation}} + \underbrace{R}_{\text{rotation}} \vec{p}$$

where $\cdot$ defines an operation that goes from $(\vec{d}, R)$ and $\vec{p}$ to $\vec{p}\,'$.

(We will drop the vector hats ($\vec{\cdot}$) on the points for the sake of simplicity)

Considering only single transformations, we can obtain:

$$p_C \text{ in frame B is } p_C^B = \vec{d}_{BC}^B + R(\theta_{BC})\vec{p}$$
$$p_B \text{ in frame A is } p_B^A = \vec{d}_{AB}^A + R(\theta_{AB})\vec{p}$$

But what if we want to find p_C in frame A? We can use the following relationship:

$$\begin{aligned} p_C^A &= \vec{d}_{AB}^A + R(\theta_{AB}) p_C^B \\ &= \vec{d}_{AB}^A + R(\theta_{AB}) \left(\vec{d}_{BC}^B + R(\theta_{BC})\vec{p} \right) \\ &= \underbrace{\vec{d}_{AB}^A + R(\theta_{AB})\vec{d}_{BC}^B}_{\vec{d}_{AC}^A} + \underbrace{R(\theta_{AB})R(\theta_{BC})}_{R(\theta_{AC}} \vec{p} \end{aligned}$$

We can write this multiple transformation in vector coordinate form as:

$$\begin{aligned} g_{AC} &= g_{AB} \cdot g_{BC} \\ &= (\vec{d}_{AB}^A, R(\theta_{AB})) \cdot (\vec{d}_{BC}^B, R(\theta_{BC})) \\ &= (\vec{d}_{AB}^A + R(\theta_{AB})\vec{d}_{BC}^B, R(\theta_{AB})R(\theta_{BC})) \\ &= (\vec{d}_{AC}^A, R(\theta_{AC})) \end{aligned}$$

Product Structure of Transformations

In summary, if a rigid body undergoes two displacements g_1 and g_2, then the total displacement g and the individual displacements are related by:

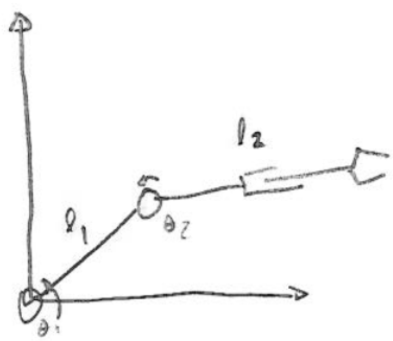
$$g = g_1 \cdot g_2 = (\vec{d}_1, R_1) \cdot (\vec{d}_2, R_2) = (\vec{d}_1 + R_1\vec{d}_2, R_1R_2)$$

Note: order matters!

$$(\vec{d}_2, R_2) \cdot (\vec{d}_1, R_1) = (\vec{d}_2 + R_1\vec{d}_1, R_2R_1) \neq$$
$$(\vec{d}_1, R_1) \cdot (\vec{d}_2, R_2) = (\vec{d}_1 + R_1\vec{d}_2, R_1R_2)$$

Example

Let's consider an example that applies these concepts to manipulation. Specifically, consider the planar robot shown in the figure below. This robot has 2 rotary joints and two links.



Question: What is the end-effectors configuration in reference to the origin frame?

$$g_e = \begin{Bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) \\ \theta_1+\theta_2 \end{Bmatrix} \qquad \text{(vector form of } x, y, \theta\text{)}$$

But there's a more programmatic way to do this.

$$g_1^0 = \left(\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, R(\theta_1) \right)$$
$$g_2^1 = \left(\begin{Bmatrix} l_1 \\ 0 \end{Bmatrix}, R(\theta_2) \right)$$
$$g_3^2 = \left(\begin{Bmatrix} l_2 \\ 0 \end{Bmatrix}, I \right)$$

We can then compute the product of these ransformations:

$$\begin{aligned} g_e &= g_1^0 \cdot g_2^1 \cdot g_3^2 \\ &= (d_1, R_1) \cdot (d_2, R_2) \cdot (d_3, R_3) \\ &= (0, R_1) \cdot (d_2, R_2) \cdot (d_3, I) \\ &= (R_1 d_2, R_1 R_2) \cdot (d_3, I) \\ &= (R_1 d_2 + R_1 R_2 d_3, R_1 R_2) \\ &= \left\{ \begin{matrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \\ \theta_1 + \theta_2 \end{matrix} \right\} \end{aligned}$$

Inverse Transformation

Now that we have a product structure for transformations, we derive how to apply an inverse transformation.

First, we must start with deriving an identity transformation. We will do this by solving for the transformation e such that $e \cdot g = g$:

$$\begin{aligned} (\vec{d}_{\mathrm{e}}, R_e) \cdot (\vec{d}, R) &= (\vec{d}, R)? \\ (\vec{d}_{\mathrm{e}} + R_e \vec{d}, R_e R) &= (\vec{d}, R) \end{aligned}$$

To have this be true, it must mean the following:

$$\begin{aligned} R_e R &= R \\ &\implies R_e = \mathbb{1} \end{aligned}$$

and thus,

$$\begin{aligned} \vec{d}_{\mathrm{e}} + R_e \vec{d} &= \vec{d} \\ \vec{d}_{\mathrm{e}} + \mathbb{1} \vec{d} &= \vec{d} \\ &\implies d_e = 0 \end{aligned}$$

Thus, an identity transformation is $e = (0, \mathbb{1})$.

Now we can use this identity transformation e to derive the form of an inverse transformation.

Specifically, we will solve for the inverse transformation g^{-1} that satisfies the relationship $(g^{-1} \cdot g = e)$:

$$\begin{aligned} (\vec{d}_{\mathrm{i}}, R_i) \cdot (\vec{d}, R) &= (0, \mathbb{1}) \\ (\vec{d}_{\mathrm{i}} + R_i \vec{d}, R_i R) &= (0, \mathbb{1}) \end{aligned}$$

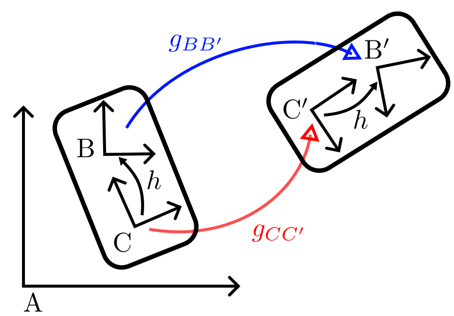
For this equation to be true, it must hold that:

$$R_i R = \mathbb{1} \implies R_i = R^{-1}$$
$$\vec{d}_{\mathrm{i}} + R_i \vec{d} = 0 \implies \vec{d}_{\mathrm{i}} = -R_i \vec{d} = -R^{-1}\vec{d}$$

Thus, the inverse transformation is $g_i = (-R^{-1}\vec{d}, R^{-1})$ and is denoted by g^{-1}. Note that this transformation aligns with the inverse rotation element from our $SO(2)$ group, $R^{-1} = R^T$.

Example

Let's consider the following example for how to use an inverse transformation.



An interpretation of the diagram above would be you sitting at a table with a friend. Your perspective is coordinate frame B, and your friend's perspective is coordinate frame C. We're then going to move the table.

Assume that you know how your position moves $g_{\mathrm{BB}'}$ and you know the transformation between you and your friend (h).

Question: How can you solve for the displacement of your friend with respect to their own reference frame (i.e., $g_{\mathrm{CC}'}$)?

Can we use these operations to understand how to change reference frame of a displacement? Let's consider the following example:

Answer: follow the arrows

1. Start at frame C
2. Apply transformation h
3. Apply transformation $g_{\mathrm{BB}'}$
4. Apply the inverse transformation h^{-1}

Together this results in the overall transformation:

$$g_{\mathrm{CC}'} = h \cdot g_{\mathrm{BB}'} \cdot h^{-1}$$

This operation is formally termed the *Adjoint Operation* and is defined as follows:

$$\mathrm{Ad}_h g = hgh^{-1} \qquad \text{(this is implicitly } h \cdot g \cdot h^{-1}\text{, but we will drop } \cdot \text{ from now on)}$$

The adjoint operation $\mathrm{Ad}_h g$ is conducted when we want to change the coordinate frame of a transformation g by the transformation h.