Topics Covered:

- Manipulator Types
- Examples

Additional Reading:

- LP 5.3;

Review

So far, we've introduced manipulator Jacobians and several ways to compute them. Today we will be discussing how the Jacobian relates to singularities for various manipulator "types".

Manipulator Types (based on Workspace)

Note: this description of manipulator types is more functional, less geometric than before.

Let $n =$ dimension of task space, $m =$ dimension of joint space. In this case, $J \in \mathbb{R}^{n \times m}$.

The three (functional) classes to describe a given manipulator are:

1. kinematically insufficient ($n > m$)
   - can get a "better" manipulator so that $m = n$.
   - can reduce dimension of task space by redefining what the task is.
2. kinematically redundant ($n < m$)
   - have more degrees of freedom than needed (Inverse kinematics have infinite solutions).
   - it's tricky, but this case can be dealt with.
3. kinematically sufficient ($n = m$)
   - need to ensure operation in dextrous workspace for maximal mobility/control.
   - equivalently, need to be away from singularities (will discuss this now)

Definition: Singularities

A **singular configuration** or **singularity** is a joint configuration of an open-chain manipulator in which the end-effector instantaneously loses a degree of freedom of its motion capability versus the number of degrees of freedom that normally prevail (i.e., the end-effector loses its ability ot move in one or more directions).

Note here that open-chain means that joint-link-joint... combinations do *not* form a loop.

Practical implications of singularities:

- manipulator loses effectiveness
- high joint velocities may be needed near a singular configuration in order to track a specified trajectory
- the manipulator can have high mechanical advantage in a singular configuration. (bench pressing in a gym as your arm is fully extended)

Finding singular configurations:

To find singularities, we can examine the manipulator Jacobian:

1. Numerical Approach

   $$\xi = J(\theta)\dot{\theta}$$

   Test the rank of J (the number of linearly independent columns)

2. For kinematically sufficient arms,

   $$\det(J(\theta)) = 0$$

   at a singular configuration.

3. For redundant arms ($n < m$), $J(\theta)J^\top(\theta)$ has the same rank as $J(\theta)$ but is square. That means a perfectly valid test is:

   $$\det(J(\theta)J^\top(\theta)) = 0$$

   for singular configurations.

4. For kinematically insufficient arms ($n > m$), $J^\top(\theta)J(\theta)$ has the same rank as $J(\theta)$ but is square. Thus, a test for singular configurations is $\det(J^\top J) = 0$.

Note, that for condition 4, the old notes says that there is an "issue" with this point, because one should not really even have this case in practice. If an arm is insufficient, it is best to get a new arm or modify the task to be sufficient. Also, the product $J^\top J$ has nonsense units if dimensional analysis is performed.

To find the singular configurations means applying one of the given tests for all θ in the joint space and keeping track of the singular ones.

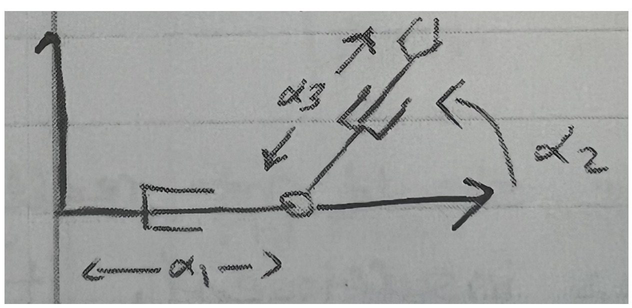
There is connection between loss of rank in the Jacobian, and loss of control.

There are usually two occasions when a singularity occurs:

1. Workspace - bounded singularities
   - when manipulator is fully extended or folded back on itself in such a way that the end-effector is at, or near, the workspace boundary
2. Workspace - interior singularities
   - occur away from workspace boundary, generally caused by a lining up of two or more axes.

Example 1

Consider the following manipulator:



This manipulator has the forward kinematics (using product of Lie groups):

$$g_e(\alpha) = g_1(\alpha_1) g_2(\alpha_2) g_3(\alpha_3)$$

with

$$g_1 = \begin{bmatrix} I & \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} R(\alpha_2) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} I & \begin{bmatrix} \alpha_3 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Once multiplied, this gives us:

$$g_e(\alpha) = \begin{bmatrix} R(\alpha_2) & \begin{bmatrix} \alpha_1 + \alpha_3 \cos(\alpha_2) \\ \alpha_3 \sin(\alpha_2) \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

To analyze the singularities, we will compute the body manipulator Jacobian. But it should be noted that the definition of a singularity is independent of choice of body or space Jacobian.

To compute body manipulator Jacobian, we will use the formula:

$$J^b = \begin{bmatrix} \mathrm{Ad}^{-1}_{g_2 g_3} J^b_1 & \mathrm{Ad}^{-1}_{g_3} J^b_2 & J^b_3 \end{bmatrix}$$

with

$$J^b_i = \left(g_i^{-1} \frac{\partial g_i}{\partial \alpha_i} \right)^{\vee}$$

Computing each body Jacobian gives us:

$$\begin{aligned} J^b_1(\alpha_1) &= \left(g_1^{-1} \frac{\partial g_1}{\partial \alpha_1} \right)^{\vee} \\ &= \left(\begin{bmatrix} I & -\begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix} \right)^{\vee} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} J^b_2(\alpha_2) &= \left(g_2^{-1} \frac{\partial g_2}{\partial \alpha_2} \right)^{\vee} \\ &= \left(\begin{bmatrix} R^{\top}(\alpha_2) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(\alpha_2) & -\cos(\alpha_2) & 0 \\ \cos(\alpha_2) & -\sin(\alpha_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{\vee} \\ &= \begin{bmatrix} \cancel{-\cos(\alpha_2)\sin(\alpha_2) + \sin(\alpha_2)\cos(\alpha_2)} & -\cos(\alpha_2)^2 - \sin(\alpha_2)^2 & 0 \\ \sin(\alpha_2)^2 + \cos(\alpha_2)^2 & \cancel{\sin(\alpha_2)\cos(\alpha_2) - \cos(\alpha_2)\sin(\alpha_2)} & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\vee} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
J_3^b &= \left(g_3^{-1} \frac{\partial g_3}{\partial \alpha_3} \right)^\vee \\
&= \begin{bmatrix} I & -\begin{bmatrix} \alpha_3 \\ 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^\vee \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Next, accounting for the adjoints:

$$
\begin{aligned}
\mathrm{Ad}_{g_2 g_3}^{-1} J_1^b &= (g_2 g_3)^{-1} J_1^b (g_2 g_3) \\
&= g_3^{-1} g_2^{-1} J_1^b g_2 g_3 \\
&= \underbrace{\underbrace{\underbrace{\underbrace{\begin{bmatrix} I & -\begin{bmatrix} \alpha_3 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^\top(\alpha_2) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}}_{\begin{bmatrix} R^\top(\alpha_2) & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 & \cos(\alpha_2) \\ 0 & 0 & -\sin(\alpha_2) \\ 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} R(\alpha_2) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 & \cos(\alpha_2) \\ 0 & 0 & -\sin(\alpha_1) \\ 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} I & \begin{bmatrix} \alpha_3 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 & \cos(\alpha_2) \\ 0 & 0 & -\sin(\alpha_1) \\ 0 & 0 & 0 \end{bmatrix}^\vee}{}^\vee \\
&= \begin{bmatrix} \cos(\alpha_2) \\ -\sin(\alpha_2) \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ad}_{g_3}^{-1} J_2^b &= g_3^{-1} J_2^b g_3 \\
&= \begin{bmatrix} I & -\begin{bmatrix}\alpha_3 \\ 0\end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & \begin{bmatrix}\alpha_3 \\ 0\end{bmatrix} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \alpha_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & \alpha_3 \\ 0 & 0 & 0 \end{bmatrix}^\vee = \begin{bmatrix} 0 \\ \alpha_3 \\ 1 \end{bmatrix}
\end{aligned}
$$

Note that a shortcut for this is $\mathrm{Ad}_g = \begin{bmatrix} R & 0 \\ [d]_\times R & R \end{bmatrix}$

Putting everything together gives us:

$$
J(\alpha) = \begin{bmatrix} \cos(\alpha_2) & 0 & 1 \\ -\sin(\alpha_2) & \alpha_3 & 0 \\ 0 & 1 & 0 \end{bmatrix}
$$

Note that an equivalent way to obtain this jacobian would have been:

$$
J^b = \begin{bmatrix} \mathrm{Ad}^{-1}_{e^{\xi_1\theta_1}\cdots e^{\xi_3\theta_3} g_0}\xi_1 & \mathrm{Ad}^{-1}_{e^{\xi_2\theta_2}\cdots e^{\xi_3\theta_3} g_0}\xi_2 & \mathrm{Ad}^{-1}_{e^{\xi_3\theta_3} g_0}\xi_3 \end{bmatrix}
$$

with

$$
\xi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

Example MATLAB Code

```
% Helper functions
twist = @(w, p) [cross(-w,p); w];
skew = @(w) [0 -w(3) w(2); w(3) 0 -w(1); -w(2) w(1) 0];
unskew = @(skw) [skw(3,2);skw(1,3);skw(2,1)];
hat = @(twist) [skew(twist(4:6)), twist(1:3); 0 0 0 0];
vee = @(htwist) [htwist(1:3,4); unskew(htwist(1:3,1:3))];
Rz = @(theta) [cos(theta) -sin(theta) 0; ...
                sin(theta) cos(theta) 0; 0 0 1];
Rx = @(theta) [1 0 0; 0 cos(theta) -sin(theta); ...
                0 sin(theta) cos(theta)];

% Symbolic variables
syms a1 a2 a3

% Specify exponential mapping twists for zero configuration
twist1 = [1; 0; 0; 0; 0; 0];
twist2 = [0; 0; 0; 0; 0; 1];
twist3 = [1; 0; 0; 0; 0; 0];
g0 = [eye(3), [0;0;0]; 0 0 0 1];

%Compute body twists using method of adjoints
twistb1 = vee(inv(expm(hat(twist1)*a1)*expm(hat(twist2)*a2)* ...
            expm(hat(twist3)*a3)*g0)* ...
            hat(twist1)* ...
            expm(hat(twist1)*a1)*expm(hat(twist2)*a2)* ...
            expm(hat(twist3)*a3)*g0)
twistb2 = vee(inv(expm(hat(twist2)*a2)*expm(hat(twist3)*a3)*g0)* ...
            hat(twist2)* ...
            expm(hat(twist2)*a2)*expm(hat(twist3)*a3)*g0)
twistb3 = vee(inv(expm(hat(twist3)*a3)*g0)* ...
            hat(twist3)*expm(hat(twist3)*a3)*g0)

% Simplify body manipulator jacobian
J = simplify([twistb1, twistb2, twistb3])
```

We are going to evaluate the body manipulator Jacobian for two cases:

Case 1: $\alpha = (1, \pi/2, 1)^\top$

$$J^b(\alpha) = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Matrix has full rank (number of linearly independent columns = 3). Thus, we have full (local) control.

Case 2: $\alpha = (1, 0, 1)^\top$

$$J^b(\alpha) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Matrix is *not* full rank (lost control of the system locally).

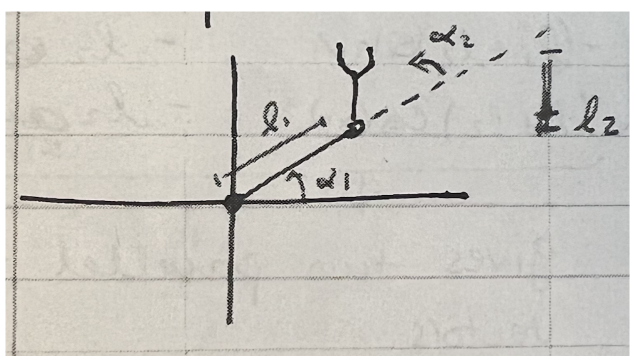
General case: $\alpha = (1, \alpha_2, 1)^\top$

$$J^b(\alpha) = \begin{bmatrix} \cos(\alpha_2) & 0 & 1 \\ -\sin(\alpha_2) & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The matrix loses rank for $\alpha_2 = \{0, \pi, \cdots 2\pi k\}$. We can also look at the determinant, $\det(J^b(\alpha)) = -\sin(\alpha)$, which approaches 0 as we approach the singularities.

Example 2

Now lets consider a slightly different example:



In this example, our homogeneous transformation matrices are:

$$g_1(\alpha_1) = \begin{bmatrix} R(\alpha_1) & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad g_2(\alpha_2) = \begin{bmatrix} R(\alpha_2) & \begin{bmatrix} l_1 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad g_3(\alpha_3) = \begin{bmatrix} I & \begin{bmatrix} l_2 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Putting these together, the forward kinematics is given by:

$$\begin{aligned} g_e(\alpha) &= g_1(\alpha_1) g_2(\alpha_2) g_3(\alpha_3) \\ &= \begin{bmatrix} R(\alpha_1 + \alpha_2) & \begin{bmatrix} l_1 \cos(\alpha_1) + l_2 \cos(\alpha_1 + \alpha_2) \\ l_1 \sin(\alpha_1) + l_2 \sin(\alpha_1 + \alpha_2) \end{bmatrix} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

This is a kinematically insufficient arm ($m < n$, $2 < 3$). This means that we can't expect total control. One solution is to only consider the outputs being x and y coordinates. This reduces the forward kinematics to:

$$p_e(\alpha) = \begin{bmatrix} l_1\cos(\alpha_1) + l_2\cos(\alpha_1+\alpha_2) \\ l_1\sin(\alpha_1) + l_2\sin(\alpha_1+\alpha_2) \end{bmatrix}$$

Since we deal with $p_e(\alpha)$ only and not $g_e(\alpha)$, there is no orientation to worry about, therefore, the standard Jacobian will be sufficient for our singularity analysis:

$$J_{pos}(\alpha) = \frac{\partial p_e(\alpha)}{\partial \alpha} = \begin{bmatrix} -l_1\sin(\alpha_1) - l_2\sin(\alpha_1+\alpha_2) & -l_2\sin(\alpha_1+\alpha_2) \\ l_1\cos(\alpha_1) + l_2\cos(\alpha_1+\alpha_2) & l_2\cos(\alpha_1+\alpha_2) \end{bmatrix}$$

Lets consider the following two cases:

Case 1: $\alpha = (\alpha_1, 0)^\top$

$$J(\alpha) = \begin{bmatrix} -(l_1+l_2)\sin(\alpha_1) & -l_2\sin(\alpha_1) \\ (l_1+l_2)\cos(\alpha_1) & -l_2\cos(\alpha_1) \end{bmatrix}$$

This gives two parallel translation vectors. $J(\alpha_1, 0)$ does not have full rank (this is a singularity).

Case 2: $\alpha = (\alpha_1, \pi/2)^\top$

$$J(\alpha) = \begin{bmatrix} -l_1\sin(\alpha_1) - l_2\cos(\alpha_1) & -l_2\sin(\alpha_1) \\ l_1\cos(\alpha_1) - l_2\sin(\alpha_1) & l_2\cos(\alpha_1) \end{bmatrix}$$

This gives two non-parallel translation vectors. so $J(\alpha_1, \pi/2)$ has full rank (no singularity).

Note: the simplification of $J(\alpha)$ above comes from the identities:

$$\sin(\alpha_1+\alpha_2) = \sin(\alpha_1)\cos(\alpha_2) + \cos(\alpha_1)\sin(\alpha_2)$$
$$\cos(\alpha_1+\alpha_2) = \cos(\alpha_1)\cos(\alpha_2) - \sin(\alpha_1)\sin(\alpha_2)$$