Topics Covered:

- Review
- Example
- Derivation for Product of Homogeneous Transformation Matrices
- Derivation for Product of Exponentials

Additional Reading:

- LP Chapter 5; MLS Chapter 3, Section 4

Review

Last lecture, we introduced the manipulator jacobian which maps joint velocities to the Lie algebra of the end-effector frame relative to either the fixed frame (spatial frame) or the end-effector frame (body frame):

$$\begin{aligned}
\hat{\xi}^s &= J^s(\theta)\dot{\theta} \\
&= \begin{bmatrix} \xi_1 & \xi_2' & \cdots & \xi_n' \end{bmatrix} \dot{\theta} \\
&= \begin{bmatrix} \left(\frac{\partial g_e}{\partial \theta_1} g_e^{-1} \right)^\vee & \cdots & \left(\frac{\partial g_e}{\partial \theta_n} g_e^{-1} \right)^\vee \end{bmatrix} \dot{\theta} \\
&= \begin{bmatrix} \xi_1 & \mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1}} \xi_2 & \cdots & \mathrm{Ad}_{\prod_{j=1}^{n-1} e^{\hat{\xi}_j \theta_j}} \xi_n \end{bmatrix} \dot{\theta} \\
\hat{\xi}^b &= J^b(\theta)\dot{\theta} \\
&= \begin{bmatrix} \xi_1^\dagger & \cdots & \xi_n^\dagger \end{bmatrix} \dot{\theta} \\
&= \begin{bmatrix} \left(g_e^{-1} \frac{\partial g_e}{\partial \theta_1} \right)^\vee & \cdots & \left(g_e^{-1} \frac{\partial g_e}{\partial \theta_n} \right)^\vee \end{bmatrix} \dot{\theta} \\
&= \begin{bmatrix} \mathrm{Ad}^{-1}_{\prod_{j=1}^{n} e^{\hat{\xi}_j \theta_j} g_0} \xi_1 & \cdots & \mathrm{Ad}^{-1}_{e^{\hat{\xi}_n \theta_n} g_0} \xi_n \end{bmatrix} \dot{\theta}
\end{aligned}$$

In the rest of today's lecture we will derive these expressions, as well as slightly different expressions that use the following "body terms":

$$\begin{aligned}
J_i^b(\theta_i) &= g_i^{-1} \frac{\partial g_i}{\partial \theta_i}, \\
J_i^s(\theta_i) &= \mathrm{Ad}_{g_i} J_i^b(\theta_i) = \frac{\partial g_i}{\partial \theta_i} g_i^{-1}.
\end{aligned}$$

The advantage of these terms is that the partial derivatives are easier to obtain in practice.

Uses of Manipulator Jacobians

The manipulator Jacobian is a key tool in robotics for a variety of applications including inverse kinematics, path planning, force control, and singularit/workspace analysis. The key equations that are used with Jacobians are the following.

For inverse kinematics, the Newton-Raphson method uses the iterative update law:

$$\theta_{k+1} = \theta_k + \underbrace{(\frac{\partial g}{\partial \theta}(\theta_k))^{-1}(g(\theta_k))}_{\delta(\theta_k)}$$
$$\delta(\theta) = J^{-1}(\theta_0)(x_d - f(\theta_0))$$

Here, if $J \in \mathbb{R}^{m \times n}$ is not square, we will use the pseudo-inverse:

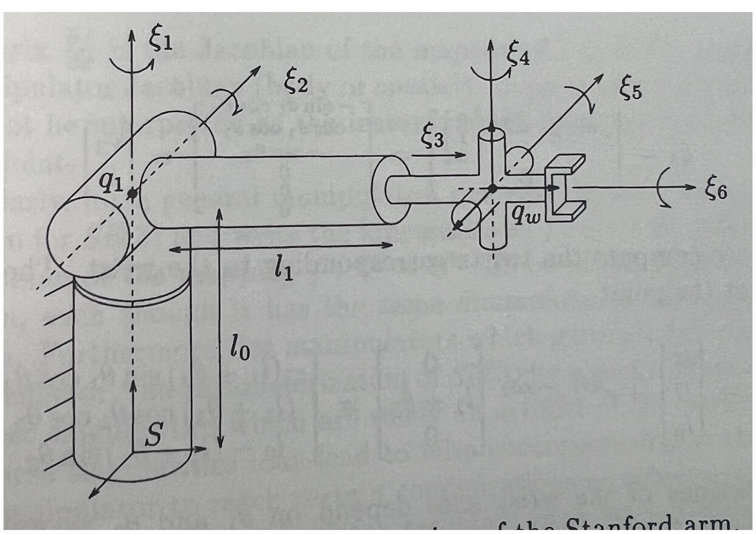
$$J^{-1} \approx J^{\dagger} := \begin{cases} J^T(JJ^T)^{-1} & \text{if } J \text{ is fat } (n > m) \\ (J^TJ)^{-1}J^T & \text{if } J \text{ is tall } (n < m) \end{cases}$$

The process of solving for joint velocities is a bit simpler:

$$\dot{\theta} = J^{-1}(\theta)\xi_d$$

Example

Consider the following manipulator:



Lets first calculate the forward-kinematics using the product of exponentials (just for practice and comparison purposes):

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\ l_0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi_3 = \begin{bmatrix} v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (v_i \text{ is a unit vector pointing along the translational axis})$$

$$\xi_4 = \begin{bmatrix} -\omega_4 \times q_4 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} l_1 + \theta_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\xi_5 = \begin{bmatrix} -\omega_5 \times q_5 \\ \omega_5 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\ -l_0 \\ l_1 + \theta_3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi_6 = \begin{bmatrix} -\omega_6 \times q_6 \\ \omega_6 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} -l_0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Lastly, our g_0 is:

$$g_0 = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Thus, our forward kinematics are:

$$g_e = e^{\xi_1 \theta_1} \cdots e^{\xi_6 \theta_6} g_0 \tag{1}$$

Now, let's compute the spatial manipulator Jacobian for this manipulator using the formula:

$$J^s = \begin{bmatrix} \xi_1 & \xi_2' & \xi_3' & \xi_4' & \xi_5' & \xi_6' \end{bmatrix}$$

For the first two joint axes, the point along the axes does not change with joint angles (i.e., $q_1 = [0, 0, l_0]^\top$). However, ω_2 will now be:

$$\omega_2' = R_z(\theta_1) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos(\theta_1) \\ -\sin(\theta_1) \\ 0 \end{bmatrix}$$

Also, the prismatic joint will be:

$$v_3' = R_z(\theta_1) R_x(-\theta_2) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -s_1 c_2 \\ c_1 c_2 \\ -s_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Next, the wrist location (q_w') will be:

$$q_w' = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} + R_z(\theta_1) R_x(-\theta_2) \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -(l_1 + \theta_3) s_1 c_2 \\ (l_1 + \theta_3) c_1 c_2 \\ l_0 - (l_1 + \theta_3) s_2 \end{bmatrix}$$

Lastly, the final three joint axes will be:

$$\begin{aligned}
\omega_4' &= R_z(\theta_1) R_x(-\theta_2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_1 s_2 \\ c_1 s_2 \\ c_2 \end{bmatrix} \\
\omega_5' &= R_z(\theta_1) R_x(-\theta_2) R_z(\theta_4) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1 c_4 + s_1 c_2 s_4 \\ -s_1 c_4 - c_1 c_2 s_4 \\ s_2 s_4 \end{bmatrix} \\
\omega_6' &= R_z(\theta_1) R_x(-\theta_2) R_z(\theta_4) R_x(-\theta_5) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_5 (s_1 c_2 c_4 + c_1 s_4) + s_1 s_2 s_5 \\ -c_5 (c_1 c_2 c_4 - s_1 s_4) - c_1 s_2 s_5 \\ -s_2 c_4 c_5 - c_2 s_5 \end{bmatrix}
\end{aligned}$$

All of these updated parameters would then be plugged into the following formula for the complete spatial manipulator Jacobian:

$$J^s = \begin{bmatrix} 0 & -\omega_2' \times q_1 & v_3' & -\omega_4' \times q_w' & -\omega_5' \times q_w' & -\omega_6' \times q_w' \\ \omega_1 & \omega_2' & 0 & \omega_4' & \omega_5' & \omega_6' \end{bmatrix}$$

Derivation of Manipulator Jacobian using Homogeneous Transformation Matrices

In the rest of lecture, we will derive the following expressions:

$$\begin{aligned} J^b(\theta) &= \left[\mathrm{Ad}^{-1}_{\left(\prod_{j=2}^{n+1} g_j\right)} J_1^b(\theta_1) \quad \cdots \quad \mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{n+1} g_j\right)} J_k^b(\theta_k) \quad \cdots \quad \mathrm{Ad}^{-1}_{g_{n+1}} J_n^b(\theta_n)\right] \\ J^s(\theta) &= \left[J_1^s(\theta_1) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{k-1} g_j} J_k^s(\theta_k) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{n-1} g_j} J_n^s(\theta_n)\right] \end{aligned}$$

$$\begin{aligned} J^b(\theta) &= \left[\mathrm{Ad}^{-1}_{\prod_{j=2}^{n}\left(e^{\xi_j\theta_j}\right) g_0} J_1^b \quad \cdots \quad \mathrm{Ad}^{-1}_{\prod_{k+1}^{n}\left(e^{\xi_j\theta_j}\right) g_0} J_k^b \quad \cdots \quad \mathrm{Ad}^{-1}_{g_0} J_n^b\right] \\ J^s(\theta) &= \left[J_1^s(\theta) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{k-1} e^{\xi_j\theta_j}} J_k^s(\theta_k) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{n-1} e^{\xi_j\theta_j}} J_n^s(\theta)\right] \end{aligned}$$

where each individual body term is defined as:

$$\begin{aligned} J_i^b &= g_i^{-1}\frac{\partial g_i}{\partial \theta_i}, \\ J_i^s &= \mathrm{Ad}_{g_i} J_i^b = \frac{\partial g_i}{\partial \theta_i} g_i^{-1}. \end{aligned}$$

Let $\theta = \theta(t)$ be the joint configuration of a manipulator. The end-effector trajectory is then $g_e(\theta(t))$.

Taking the time derivative, we get:

$$\begin{aligned} \dot{g}_e &= \frac{d}{dt}\left[g_e(\theta(t))\right] \\ &= \frac{\partial g_e}{\partial \theta}\frac{\partial \theta}{\partial t} \\ &= Dg_e(\theta(t))\dot{\theta}(t) \\ &= \begin{bmatrix} | & & | \\ \frac{\partial g_e}{\partial \theta_1} & \cdots & \frac{\partial g_e}{\partial \theta_n} \\ | & & | \end{bmatrix} \dot{\theta}(t) \end{aligned}$$

Let's think about what these partial derivatives ($\frac{\partial g_e}{\partial \theta_i}$) mean when the representation for g_e is in homogeneous coordinates?

First, let's switch frames since $\dot{g}_e = Dg_e\dot{\theta}$ has mixed frames.

- Body Frame: $J^b(\theta) = g_e^{-1}(\theta) \cdot \frac{\partial g_e(\theta)}{\partial \theta}$ (this is the body manipulator Jacobian)
- Spatial Frame: $J^s(\theta) = \frac{\partial g_e(\theta)}{\partial \theta} \cdot g_e^{-1}(\theta)$ (this is the spatial manipulator Jacobian)

In mixed frames, $J(\theta) = \frac{\partial g_e(\theta)}{\partial \theta}$ is called the *manipulator Jacobian*.

In practice, we work in homogeneous form where needed or when the computations are simpler, then convert to vector form by unhatting the result.

To do this derivation, let's consider a basic example that only has 2 joints and 2 links, with the joints located at the start of the first and second link:

$$g_e(\theta) = g_1(\theta_1) g_2(\theta_2) g_3$$

To compute the body manipulator Jacobian, we would do:

$$\begin{aligned}
J^b(\theta) &= g_e^{-1}(\theta) \cdot \frac{\partial g_e(\theta)}{\partial \theta} \\
&= g_e^{-1}(\theta) \begin{bmatrix} | & | \\ \frac{\partial g_e}{\partial \theta_1} & \frac{\partial g_e}{\partial \theta_2} \\ | & | \end{bmatrix} \\
&= g_e^{-1}(\theta) \left[\tfrac{\partial g_1}{\partial \theta_1} g_2 g_3 \quad g_1 \tfrac{\partial g_e}{\partial \theta_2} g_3 \right] \\
&= g_3^{-1} g_2^{-1} g_1^{-1} \left[\tfrac{\partial g_1}{\partial \theta_1} g_2 g_3 \quad g_1 \tfrac{\partial g_e}{\partial \theta_2} g_3 \right] \\
&= \left[g_3^{-1} g_2^{-1} \underbrace{g_1^{-1} \frac{\partial g_1}{\partial \theta_1}}_{\text{body term}} g_2 g_3 \quad g_3^{-1} \underbrace{g_2^{-1} \cancel{g_1^{-1} g_1} \frac{\partial g_e}{\partial \theta_2}}_{\text{body term}} g_3 \right]
\end{aligned}$$

Let's look at these two body terms in particular:

$$J_1^b = g_1^{-1} \frac{\partial g_1}{\partial \theta_1} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(\theta_1) & -\cos(\theta_1) & 0 \\ \cos(\theta_1) & -\sin(\theta_1) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$J_2^b = g_2^{-1} \frac{\partial g_2}{\partial \theta_2} = \begin{bmatrix} \cos(\theta_2) & \sin(\theta_2) & -\cos(\theta_2) l_1 \\ -\sin(\theta_2) & \cos(\theta_2) & -\sin(\theta_2) l_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(\theta_2) & -\cos(\theta_2) & 0 \\ \cos(\theta_2) & -\sin(\theta_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Plugging these terms back in:

$$\begin{aligned}
J^b(\theta) &= \left[g_3^{-1} g_2^{-1} J_1^b g_2 g_3 \quad g_3^{-1} J_2^b g_3 \right] \\
&= \left[\mathrm{Ad}_{g_3^{-1}} \mathrm{Ad}_{g_2^{-1}} J_1^b \quad \mathrm{Ad}_{g_3^{-1}} J_2^b \right] \\
&= \left[\mathrm{Ad}_{g_3^{-1} g_2^{-1}} J_1^b \quad \mathrm{Ad}_{g_3^{-1}} J_2^b \right] \\
&= \left[\mathrm{Ad}_{(g_2 g_3)^{-1}} J_1^b \quad \mathrm{Ad}_{g_3^{-1}} J_2^b \right]
\end{aligned}$$

This can be written as the general expression:

$$J^b(\theta) = \left[\mathrm{Ad}^{-1}_{\left(\prod_{j=2}^{n+1} g_j\right)} J^b_1(\theta_1) \quad \cdots \quad \mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{n+1} g_j\right)} J^b_k(\theta_k) \quad \cdots \quad \mathrm{Ad}^{-1}_{g_{n+1}} J^b_n(\theta_n)\right]$$

In the example, we would get the body velocity of the end-effector as a function of joint velocities $\dot{\theta}$ using:

$$\xi^b = J^b(\theta)\dot{\theta}$$

$$\xi^b = \mathrm{Ad}_{(g_2 g_3)^{-1}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \dot{\theta}_1 + \mathrm{Ad}_{g_3^{-1}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \dot{\theta}_2$$

Important note: Order of implementation is irrelevant here, $(\mathrm{Ad}_h \hat{\xi})^\vee = \mathrm{Ad}_h(\hat{\xi}^\vee)$. Remember that $\mathrm{Ad}_h \hat{\xi} = h\hat{\xi}h^{-1}$. So here, $(\mathrm{Ad}_h \hat{\xi})^\vee = (h\hat{\xi}h^{-1})^\vee = \mathrm{Ad}_h \xi$. Also, Ad_h can be written as a special matrix as defined in the following lemma.

Lemma 2.13 from MLS:

Lemma. *If $\hat{\xi} \in \mathfrak{se}(3)$ is a twist with twist coordinates $\xi \in \mathbb{R}^6$, then for any $g \in SE(3)$, $g\hat{\xi}g^{-1}$ is a twist with twist coordinates $\mathrm{Ad}_g \xi \in \mathbb{R}^6$. This adjoint is defined as:*

$$Ad_g = \begin{bmatrix} R & [d]_\times R \\ 0 & R \end{bmatrix}$$

We can derive this adjoint matrix as follows:
Let $\xi^b = \begin{bmatrix} v^b \\ \omega^b \end{bmatrix}$ and $g_{sb} = \begin{bmatrix} R_{sb} & d_{sb} \\ 0 & 1 \end{bmatrix}$. Then, we can write:

$$\begin{aligned} \omega^s &= R_{sb}\omega^b \\ v^s &= -\omega^s \times d_{sb} + \dot{d}_{sb} \\ &= d_{sb} \times (R_{sb}\omega^b) + R_{sb}v^b && \text{(Since } v^b = R_{sb}^T \dot{d}_{sb}\text{)} \end{aligned}$$

$$\begin{bmatrix} v^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} R_{sb} & [d_{sb}]_\times R_{sb} \\ 0 & R_{sb} \end{bmatrix} \begin{bmatrix} v^b \\ \omega^b \end{bmatrix}$$

This adjoint matrix transforms twists from one coordinate frame to another.

Continuing with our example, to get the spatial Jacobian, we can use the adjoint of g_e with the body Jacobian:

$$J^s(\theta) = \mathrm{Ad}_{g_e} J^b(\theta)$$

However, as you may notice, we will now have a combination of adjoint terms. Specifically, the

kth column of the manipulator Jacobian is:

$$\mathrm{Ad}_{g_e(\theta)}\mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{3} g_j\right)} J_k^b$$

Looking at just the adjoint terms:

$$\mathrm{Ad}_{g_e(\theta)}\mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{3} g_j\right)} = \mathrm{Ad}_{\left(\prod_{j=1}^{3} g_j\right)}\mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{3} g_j\right)}$$

We can also rearrange the right-hand term by observing:

$$\begin{aligned}
\mathrm{Ad}^{-1}_{(g_1 g_2 g_3)}\xi &= (g_1 g_2 g_3)^{-1}\,\xi\,(g_1 g_2 g_3) \\
&= g_3^{-1} g_2^{-1} g_1^{-1} \xi \left(g_3^{-1} g_2^{-1} g_1^{-1}\right)^{-1} \\
&= \mathrm{Ad}_{\left(g_3^{-1} g_2^{-1} g_1^{-1}\right)}
\end{aligned}$$

So using this pattern with our previous adjoint terms:

$$\mathrm{Ad}_{\left(\prod_{j=1}^{3} g_j\right)}\mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{3} g_j\right)} = \mathrm{Ad}_{\left(\prod_{j=1}^{3} g_j\right)}\mathrm{Ad}_{\left(\prod_{j=3}^{k+1} g_j^{-1}\right)}$$

However, when we actually apply these adjoints, we would end up seeing a lot of terms cancel:

$$\begin{aligned}
\mathrm{Ad}_{\left(\prod_{j=1}^{3} g_j\right)}\mathrm{Ad}_{\left(\prod_{j=3}^{k+1} g_j^{-1}\right)} J_k^b &= (g_1 g_2 \cdots g_3)\left(g_3^{-1} \cdots g_{k+1}^{-1}\right) J_k^b \left(g_3^{-1} \cdots g_{k+1}^{-1}\right)^{-1} (g_1 \cdots g_3)^{-1} \\
&= (g_1 g_2 \cdots g_3)\left(g_3^{-1} \cdots g_{k+1}^{-1}\right) J_k^b \left(g_{k+1} \cdots g_3\right)\left(g_3^{-1} \cdots g_1^{-1}\right) \\
&= (g_1 \cdots g_k) J_k^b (g_k^{-1} \cdots g_1^{-1}) \\
&= \mathrm{Ad}_{\left(\prod_{j=1}^{k} g_j\right)} J_k^b
\end{aligned}$$

Thus, plugging this back into our expression for the spatial Jacobian:

$$\begin{aligned}
J^s(\theta) &= \mathrm{Ad}_{g_e(\theta)} J^b(\theta) \\
&= \begin{bmatrix} | & & | & & | \\ \mathrm{Ad}_{g_1} J_1^b & \cdots & \mathrm{Ad}_{\prod_{j=1}^{k} g_j} J_k^b & \cdots & \mathrm{Ad}_{\prod_{j=1}^{n} g_j} J_n^b \\ | & & | & & | \end{bmatrix}
\end{aligned}$$

Lastly, since we want everything in terms of the spatial frame, we can substitute in:

$$J_i^s = \mathrm{Ad}_{g_i} J_i^b$$

by pulling out the last term of the product series in the previous expression:

$$\begin{aligned}
J^s(\theta) &= \left[\mathrm{Ad}_{g_1} J_1^b \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{k-1} g_j}\mathrm{Ad}_{g_k} J_k^b \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{n-1} g_j}\mathrm{Ad}_{g_n} J_n^b\right] \\
&= \left[J_1^s(\theta_1) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{k-1} g_j} J_k^s(\theta_k) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{n-1} g_j} J_n^s(\theta_n)\right]
\end{aligned}$$

Product of Exponentials

We can repeat a very similar process for the Product of Exponentials method of forward kinematics by representing our end-effector configuration as:

$$g_e = e^{\xi_1\theta_1} \cdots e^{\xi_n\theta_n} g_0,$$

and plugging this into the formula for the body manipulator jacobian ($J^b(\theta) = g_e^{-1}\frac{\partial g_e(\theta)}{\partial\theta}$),

$$\begin{aligned}
J^b(\theta) &= \left(e^{\xi_1\theta_1} \cdots e^{\xi_n\theta_n} g_0\right)^{-1} \left[\frac{\partial\left(e^{\xi_1\theta_1}\cdots e^{\xi_n\theta_n} g_0\right)}{\partial\theta_1} \cdots \frac{\partial\left(e^{\xi_1\theta_1}\cdots e^{\xi_n\theta_n} g_0\right)}{\partial\theta_n}\right] \\
&= \left(e^{\xi_1\theta_1} \cdots e^{\xi_n\theta_n} g_0\right)^{-1} \left[\xi_1 e^{\xi_1\theta_1} \cdots e^{\xi_n\theta_n} g_0 \cdots \xi_n e^{\xi_1\theta_1} \cdots e^{\xi_n\theta_n} g_0\right] \\
&= \left(g_0^{-1} e^{-\xi_n\theta_n} \cdots e^{-\xi_1\theta_1}\right) \left[\xi_1 e^{\xi_1\theta_1} \cdots e^{\xi_n\theta_n} g_0 \cdots \left(e^{\xi_1\theta_1} \cdots \xi_n e^{\xi_n\theta_n}\right) g_0\right]
\end{aligned}$$

Here, we can further simplify these terms by substituting in the body terms:

$$\begin{aligned}
J_i^b &= g_i^{-1}\frac{\partial g_i}{\partial\theta_i} \\
&= e^{-\xi_i\theta_i}\xi_i e^{\xi_i\theta_i} = J_i^b.
\end{aligned}$$

$$\begin{aligned}
J^b(\theta) &= \left[\left(g_0^{-1} e^{-\xi_n\theta_n} \cdots e^{-\xi_2\theta_2}\right) J_1^b \left(e^{\xi_2\theta_2} \cdots e^{\xi_n\theta_n} g_0\right) \quad \cdots \quad g_0^{-1} J_n^b g_0\right] \\
&= \left[\mathrm{Ad}^{-1}_{e^{\xi_2\theta_2}\cdots e^{\xi_n\theta_n} g_0} J_1^b \quad \cdots \quad \mathrm{Ad}^{-1}_{g_0} J_n^b\right] \\
&= \left[\mathrm{Ad}^{-1}_{\prod_{j=2}^{n}\left(e^{\xi_j\theta_j}\right) g_0} J_1^b \quad \cdots \quad \mathrm{Ad}^{-1}_{\prod_{k+1}^{n}\left(e^{\xi_j\theta_j}\right) g_0} J_k^b \quad \cdots \quad \mathrm{Ad}^{-1}_{g_0} J_n^b\right]
\end{aligned}$$

Lastly, we can similarly convert this to the spatial Jacobian:

$$J^s(\theta) = \left[J_1^s(\theta) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{k-1} e^{\xi_j\theta_j}} J_k^s(\theta_k) \quad \cdots \quad \mathrm{Ad}_{\prod_{j=1}^{n-1} e^{\xi_j\theta_j}} J_n^s(\theta)\right]$$