Topics Covered:

- Introduction to Jacobians
- The Manipulator Jacobian

Additional Reading:

- LP Chapter 5

Introduction to Jacobians

The Jacobian (notice *not* the manipulator Jacobian) is used in the context of robotics to relate end-effector velocity to joint velocity as a function of joint variables.

Definition: Jacobian

Assume we have a manipulator with coordinates $x \in \mathbb{R}^m$, velocity $\dot{x} = dx/dt \in \mathbb{R}^m$, and joint variables $\theta \in \mathbb{R}^n$. The forward kinematics can be written as:

$$x(t) = f(\theta(t)).$$

Using the chain rule, the time derivative at time t is:

$$\begin{aligned}\dot{x}(t) &= \frac{d}{dt} f(\theta(t)) \\ &= \frac{\partial f}{\partial \theta} \frac{d\theta}{dt} \\ &= J(\theta)\dot{\theta}\end{aligned}$$

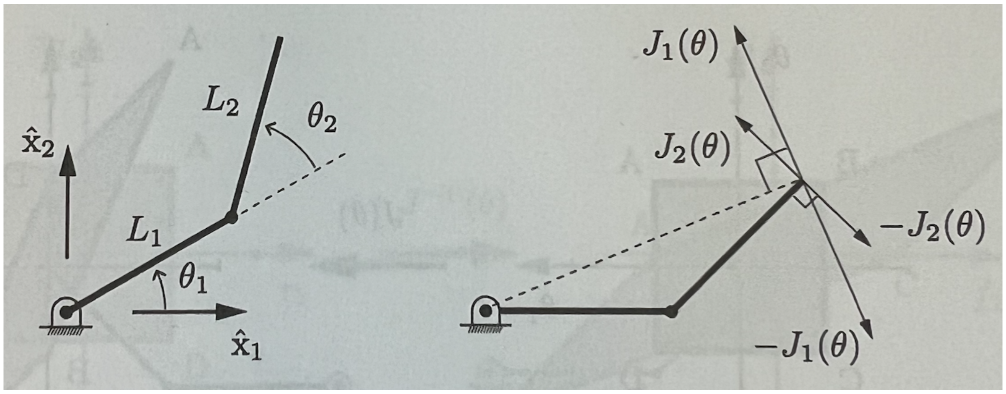
Here, $J(\theta) \in \mathbb{R}^{m \times n}$ is the Jacobian. This Jacobian matrix represents the linear sensitivity of the end-effector velocity $\dot{x}$ to the joint velocity $\dot{\theta}$ as a function of the joint variables θ. Explicitly, the Jacobian is:

$$J(\theta) := \frac{\partial f(\theta)}{\partial \theta} = Df(\theta) = \begin{bmatrix} \frac{\partial f_1(\theta)}{\partial \theta_1} & \cdots & \frac{\partial f_1(\theta)}{\partial \theta_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(\theta)}{\partial \theta_1} & \cdots & \frac{\partial f_m(\theta)}{\partial \theta_n} \end{bmatrix}$$

for $\theta = [\theta_1, \ldots, \theta_n]^\top$ and $f(\theta) = [f_1(\theta), \ldots, f_m(\theta)]^\top$.

Example

Consider the following example:



Assume that we can obtain the forward kinematics as:

$$x_1 = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2)$$
$$x_2 = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)$$

Differentiating both sides yields:

$$\dot{x}_1 = -L_1 \sin(\theta_1)\dot{\theta}_1 - L_2 \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$
$$\dot{x}_2 = L_1 \cos(\theta_1)\dot{\theta}_1 + L_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$

Rearranging this to math the form $\dot{x} = J(\theta)\dot{\theta}$, we get:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2)\end{bmatrix}\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix} \\ &= \begin{bmatrix}J_1(\theta) & J_2(\theta)\end{bmatrix}\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}\end{aligned}$$

Using this jacobian, we can write the velocity of the end-effector as $\dot{x}$:

$$v_{tip} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

We can also see that we would have obtained the same thing by directly computing $Df(\theta)$:

$$\begin{aligned}Df(\theta) &= \begin{bmatrix}\frac{\partial f_1(\theta)}{\partial \theta_1} & \frac{\partial f_1(\theta)}{\partial \theta_2} \\ \frac{\partial f_2(\theta)}{\partial \theta_1} & \frac{\partial f_2(\theta)}{\partial \theta_2}\end{bmatrix} \\ &= \begin{bmatrix}-L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2)\end{bmatrix} \\ J(\theta) &= \begin{bmatrix}J_1 & J_2\end{bmatrix}\end{aligned}$$

One special aspect of the Jacobian is that it becomes a singular matrix when $J_1(\theta)$ and $J_2(\theta)$ become collinear. THus, we can make conclusions about the *singularities* of the manipulator by looking at the Jacobian. These singularities are the configurations where the robot tip is unable to generate velocities in certain directions.

The Jacobian also relates to joint torque through the equation:

$$f_{\text{tip}}^{\top} v_{\text{tip}} = \tau^{\top} \dot{\theta}$$

which can be transformed into the expression:

$$\begin{aligned} \tau &= J^{\top}(\theta) f_{\text{tip}} \\ f_{\text{tip}} &= J^{-1}(\theta) \tau \end{aligned}$$

This notion of a *Jacobian* is a generic definition available in the literature. When applied to a manipulator, since this Jacobian is often used to express the relationship between the end-effector coordinates and the joint configuration, it is sometimes called the *coordinate* Jacobian.

The Manipulator Jacobian

A different Jacobian is the *Manipulator Jacobian*. This concept is similar, but it relates a six-dimensional twist ξ to the joint velocities. There are two standard Manipulator Jacobians, the *Spatial Manipulator Jacobian* and the *Body Manipulator Jacobian*. Each column of the spatial Jacobian corresponds to a screw axis expressed in the fixed frame with the screw axes depending on the joint variables θ. Each column of the body Jacobian corresponds to a screw axis expressed in the end-effector body frame. Note that before, for forward kinematics, our screw axes were always for the case $\theta = 0$.

We will derive how to express it with either the Product of Lie Groups or the Product of Exponentials.

Body and Spatial Manipulator Jacobians:

We will derive these expressions over the next few lectures. But in short, we have two different manipulator Jacobians of interest: the Spatial Jacobian which maps joint velocity $\dot{\theta}$ to the end-effectors spatial twist $\hat{\xi}^s$, and the Body Jacobian which maps joint velocity $\dot{\theta}$ to the end-effectors body twist $\hat{\xi}^b$. Recall that the spatial twist is the end-effector's twist expressed in fixed-frame coordinates, and the body twist is the end-effector twist expressed in the end-effector-frame coordinates.

For the following expressions, recall that the *twist* (sometimes called Lie-algebra element, sometimes called the screw) has the form:

$$\xi_i = \begin{Bmatrix} v_i \\ \omega_i \end{Bmatrix} = \begin{cases} \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix} & \text{if revolute} \\ \\ \begin{bmatrix} v_i \\ 0 \end{bmatrix} & \text{if prismatic} \end{cases}$$

The physical meaning here for ξ_i is that it describes the twist of the i^{th} joint in terms of the fixed frame with the robot in its zero position (i.e., $q_i = p_i^s(0)$). Sometimes this is also referred to as the screw-axis describing the i^{th} joint.

The *spatial manipulator Jacobian* $J^s(\theta) \in \mathbb{R}^{6\times n}$ associated with $\theta \in \mathbb{R}^n$ is defined as:

$$\begin{aligned} \hat{\xi}^s &= J^s(\theta)\dot{\theta} \\ &= \begin{bmatrix} \xi_1 & \xi_2' & \cdots & \xi_n' \end{bmatrix} \dot{\theta} \\ &= \left[\left(\frac{\partial g_e}{\partial \theta_1} g_e^{-1} \right)^\vee \quad \cdots \quad \left(\frac{\partial g_e}{\partial \theta_n} g_e^{-1} \right)^\vee \right] \dot{\theta} \\ &= \begin{bmatrix} \xi_1 & \mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1}} \xi_2 & \cdots & \mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi}_{n-1} \theta_{n-1}}} \xi_n \end{bmatrix} \dot{\theta} \end{aligned}$$

Here, ξ_i' to denote the twist evaluated at the current configuration of the robot, *not* the zero configuration (i.e., $q_i = p_i^s(\theta)$).

The *body manipulator Jacobian*, $J^b(\theta) \in \mathbb{R}^{6\times n}$ is defined as:

$$\begin{aligned} \hat{\xi}^b &= J^b(\theta)\dot{\theta} \\ &= \begin{bmatrix} \xi_1^\dagger & \xi_2^\dagger & \cdots & \xi_n^\dagger \end{bmatrix} \dot{\theta} && (\dagger \text{ is } \backslash\text{dagger}) \\ &= \left[\left(g_e^{-1} \frac{\partial g_e}{\partial \theta_1} \right)^\vee \quad \cdots \quad \left(g_e^{-1} \frac{\partial g_e}{\partial \theta_n} \right)^\vee \right] \dot{\theta} \\ &= \left[\mathrm{Ad}^{-1}_{e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi}_n \theta_n} g_0} \xi_1 \quad \cdots \quad \mathrm{Ad}^{-1}_{e^{\hat{\xi}_n \theta_n} g_0} \xi_n \right] \dot{\theta} \end{aligned}$$

Here, $\xi_i^\dagger$ are the joint twists written with respect to the tool frame at the current configuration.

These Jacobians are related by the equation:

$$J^s(\theta) = \mathrm{Ad}_{g_e(\theta)} J^b(\theta)$$

Note: The spatial velocity will be particularly useful for inverse kinematics. For example, you can compute the joint velocities required to achieve a desired end-effector velocity if J^s is invertible using the expression:

$$\dot{\theta} = J^s(\theta)^{-1} \hat{\xi}^s$$

where $\hat{\xi}^s$ is computed using the expression $\hat{\xi}^s = \dot{g}g^{-1}$. Then, by treating the expression as an ordinary differential equation for θ, we can also solve for the corresponding joint positions by integrating the equation above over some interval of time.

Also, if we want to compute the linear velocity of th end-effector, we can use the formula:

$$\dot{p}^s = \left(J^s(\theta)\dot{\theta}\right)^\wedge p^s$$
$$\dot{p}^b = \left(J^b(\theta)\dot{\theta}\right)^\wedge p^b$$

with p^s being a point attached to the frame of the end-effector, relative to the base frame, and p^b being a point attached to the frame of the end-effector, relative to the end-effector frame.

Lastly, we can still relate torques to the end-effector forces using the equation:

$$\tau = J^\top(\theta) f_{\text{tip}}$$

Example Revisted

So let's revisit the previous example. The direction of the twists are:

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then, the points on the axes (as a function of the joint variables at the current configuration and *not* the zero configuration) are:

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} L_1 \cos(\theta_1) \\ L_1 \sin(\theta_1) \\ 0 \end{bmatrix}$$

Then, the twists are:

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_2' = \begin{bmatrix} L_1 \sin(\theta_1) \\ -L_1 \cos(\theta_1) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the spatial manipulator Jacobian is:

$$\begin{aligned} J^s(\theta) &= \begin{bmatrix} \xi_1 & \xi_2' \end{bmatrix} \\ &= \begin{bmatrix} 0 & L_1 \sin(\theta_1) \\ 0 & -L_1 \cos(\theta_1) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Note that we would get the same result by using either

$$J^s = \begin{bmatrix} \xi_1 & \mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1}} \xi_2 \end{bmatrix}$$

or

$$J^s = \begin{bmatrix} \left(\frac{\partial g_e}{\partial \theta_1} g_e^{-1} \right)^\vee & \left(\frac{\partial g_e}{\partial \theta_2} g_e^{-1} \right)^\vee \end{bmatrix}$$

Lastly, as a check, we can compute the linear velocity of the end-effector using the formula:

$$\begin{bmatrix} \dot{q}^s \\ 0 \end{bmatrix} = \xi^s \begin{bmatrix} q^s \\ 1 \end{bmatrix} = (J^s \dot{\theta})^\wedge \begin{bmatrix} q^s \\ 1 \end{bmatrix}$$

where $\dot{q}^s$ is the linear velocity of point q in the fixed spatial frame (q^s).

In our example, the point of our end-effector is $p = g_0 g_1 \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$

Finally, we can solve for the linear velocity at the end-effector as:

$$\begin{aligned}
\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} &= \left(\begin{bmatrix} 0 & L_1 \sin(\theta_1) \\ 0 & -L_1 \cos(\theta_1) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \right)^\wedge \begin{bmatrix} p \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} L_1 \sin(\theta_1)\dot{\theta}_2 + L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\ L_1 \cos(\theta_1) - L_1 \cos(\theta_1)\dot{\theta}_2 + L_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

While this expression is slightly different, it will give you the same result as before!