Topics Covered:

- Rotations
- Transformations
- Exponential Coordinates
- Body and Spatial Velocity
- The Adjoint Operation
- Equations Page

Additional Reading:

- MLS Chapter 2, LP Chapter 3

Rotations

In Euclidean space, the 3x3 rotation matrix that describes the orientation of frame B with respect to frame A is given by:

$$R_{ab} = \left[x_{ab}, y_{ab}, z_{ab}\right]$$

Key properties of rotation matrices include:

$$\det R = +1 \qquad \text{(Assuming the right hand rule)}$$
$$RR^\top = R^\top R = I \qquad \text{(And thus, } R^\top = R^{-1}\text{)}$$

For rotations in the plane (about the z-axis that comes out of the page), we use the rotation matrix:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

The set of all 3x3 matrices which satisfy the properties of rotation matrices is called the Special Orthogonal Group in 3D, denoted $SO(3)$. This group satisfies the following group axioms:

1. Closure: If $R_1, R_2 \in SO(3)$, then $R_1 R_2 \in SO(3)$
2. Identity: The identity matrix is $I \in SO(3)$
3. Inverse: The inverse of $R \in SO(3)$ is $R^\top \in SO(3)$

4. Associativity: $(R_1R_2)R_3 = R_1(R_2R_3)$

Rotation matrices can be used to 1) Rotate Points, 2) Rotate Vectors, 3) Rotate Frames:

$$p_a = R_{ab}p_b \qquad (p_b = (x_b, y_b, z_b) \text{ is a point relative to frame } B)$$
$$v_a = R_{ab}v_b \qquad (v_b = q_b - p_a \text{ is some vector in frame } B)$$
$$R_{ac} = R_{ab}R_{bc} \qquad (R_{ac} \text{ is the orientation of frame } C \text{ relative to frame } A)$$

Exponential Coordinates for Rotation

We can also describe a net rotation using the exponential coordinates:

$$R(\omega, \tau) = \exp([\omega]_\times \tau)$$

where $\omega \in \mathbb{R}^3$ is a unit-vector ($\|\omega\| = 1$) specifying the direction of rotation, $\tau \in \mathbb{R}$ is the unit of time, and $[\omega]_\times$ is the skew-symmetric matrix that represents the Lie-algebra element $\mathfrak{so}(3)$ of the SO(3) group. The skew-symmetric matrix is given by:

$$[\omega]_\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

We can utilize Rodrigues' formula to solve for this exponential map:

$$\exp([\omega]_\times \tau) = I + \sin(\tau)[\omega]_\times + (1 - \cos(\tau))[\omega]_\times^2$$

Note that since t and θ are interchangeable for $p(t) = e^{[\omega]_\times t}p(0)$ or $p(\theta) = e^{[\omega]_\times \theta}p(0)$, the exponential map can also be written as $\exp([\omega]_\times \theta)$, where ω is the axis of rotation and θ is the angle of rotation.

If ω is non-unit ($\|\omega\| \neq 1$), we can utilize the general form of Rodrigues' formula:

$$\exp([\omega]_\times \tau) = I + \frac{[\omega]_\times}{\|\omega\|} \sin(\|\omega\|\tau) + \frac{[\omega]_\times^2}{\|\omega\|^2} (1 - \cos(\|\omega\|\tau))$$

If we instead wanted to solve for the Lie-algebra element (ω, τ) given a rotation matrix R, we can use the inverse of the exponential map which is called the *Logarithm*. The expression of the Logarithm for $\|\omega\| = 1$ is the following:

$$(\omega, \tau) = \ln(R)$$
$$\text{with}$$
$$\tau = \cos^{-1}\left(\frac{\text{Tr}(R) - 1}{2}\right) \qquad (\text{Tr} = r_{11} + r_{22} + r_{33})$$
$$\omega = \frac{1}{2\sin(\tau)} \begin{Bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{Bmatrix}$$

Transformations

General rigid body motion involves both translation and rotation. Thus, we describe the configuration of a rigid-body with a body frame B, relative to some frame A, as the pair (d_{ab}, R_{ab}). This configuration space is called the Special Euclidean Group in 3D, denoted $SE(3)$:

$$SE(3) = \{(d, R) \mid d \in \mathbb{R}^3, R \in SO(3)\} = \mathbb{R}^3 \times SO(3)$$

There are two ways of talking about configurations: vector form and homogeneous form. In vector form, we describe the configuration as a pair (d_{ab}, R_{ab}) where $d_{ab} \in \mathbb{R}^3$ is the translation vector and $R_{ab} \in SO(3)$ is the rotation matrix. In homogeneous form, we describe the configuration as a 4x4 matrix:

$$g_{ab} = \begin{bmatrix} R_{ab} & d_{ab} \\ 0 & 1 \end{bmatrix}$$

These transformations can be used to rigidly transform a point (p) using the expressions:

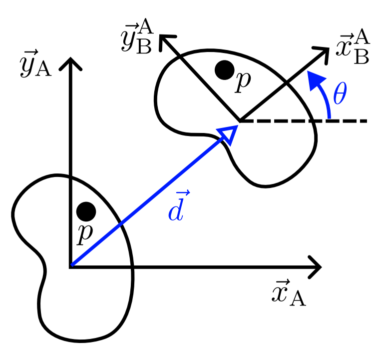
$$p' = \vec{d} + R_{ab}(\theta)p$$

or

$$p' = \begin{bmatrix} R & \vec{d} \\ 0 & 1 \end{bmatrix} p$$

These transformations can also be interpreted as representing the point p in frame A instead of on the local body frame.

This transformation is illustrated in the following diagram:



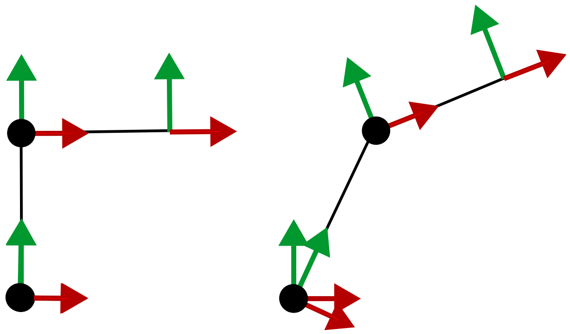
If we wanted to combine two different transformations together, there are also two ways of doing this for both the vector-form and homogeneous-form coordinates:

$$(d_{ac}, R_{ac}) = (d_{ab}, R_{ab})(d_{bc}, R_{bc}) = (d_{ab} + R_{ab}d_{bc}, R_{ab}R_{bc})$$

or

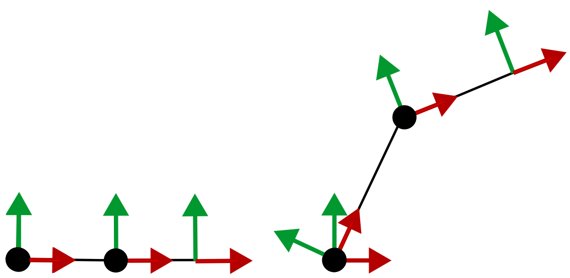
$$g_{ac} = g_{ab}g_{bc} = \begin{bmatrix} R_{ab}R_{bc} & d_{ab} + R_{ab}d_{bc} \\ 0 & 1 \end{bmatrix}$$

We can use this combination of transformations to describe the configuration of a robot manipulator. For example, let's consider the following manipulators:



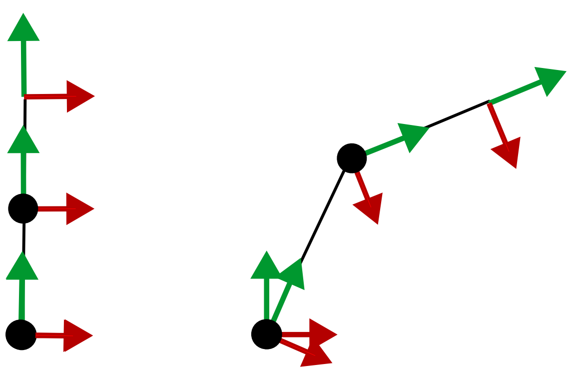
$$g_{WE} = g_{WA}g_{AB}g_{BC}g_{CE}$$
$$= \begin{bmatrix} R(\theta_1) & \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta_2) & \begin{Bmatrix} 0 \\ l_1 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(0) & \begin{Bmatrix} l_2 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

Figure 1: Example Manipulator 1: $\vec{\theta} = (\theta_1, \theta_2)$ with lengths l_1 and l_2



$$g_{WE} = g_{WA}g_{AB}g_{BC}g_{CE}$$
$$= \begin{bmatrix} R(\theta_1) & \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta_2) & \begin{Bmatrix} l_1 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(0) & \begin{Bmatrix} l_2 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

Figure 2: Example Manipulator 2: $\vec{\theta} = (\theta_1, \theta_2)$ with lengths l_1 and l_2

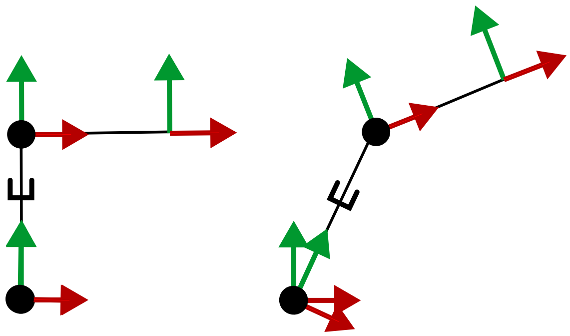


$$g_{WE} = g_{WA}g_{AB}g_{BC}g_{CE}$$
$$= \begin{bmatrix} R(\theta_1) & \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta_2) & \begin{Bmatrix} 0 \\ l_1 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(0) & \begin{Bmatrix} 0 \\ l_2 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

Figure 3: Example Manipulator 3: $\vec{\theta} = (\theta_1, \theta_2)$ with lengths l_1 and l_2

For all of these examples, note that you could derive the correct $\vec{d}$ vectors based on just the right-hand images since the frame axes are provided for you. But I've also provided you with the zero configurations on the left for clarity.

If we have robots with prismatic joints, this process is the same, we just have a variable in our $\vec{d}$ vector instead of a constant. For example, consider the following manipulator:



$$\begin{aligned} g_{WE} &= g_{WA} g_{AB} g_{BC} g_{CE} \\ &= \begin{bmatrix} R(\theta_1) & \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta_3) & \begin{Bmatrix} 0 \\ \theta_2 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(0) & \begin{Bmatrix} l_2 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Figure 4: Example Manipulator 4: $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ where θ_2 is now the prismatic joint.

Exponential Coordinates for Transformations

As with rotations, we can describe the configuration of a robot manipulator using the exponential coordinates:

$$g(\xi, \tau) = \exp([\xi]_\times \tau)$$

where here, $\xi \in \mathbb{R}^6$ is a vector that describes the twist of the manipulator and $\tau \in \mathbb{R}$ is the unit of time. The twist represents the Lie-algebra element $\mathfrak{se}(3)$ of the SE(3) group is given by:

$$\hat{\xi} = \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix}, \quad \xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$$

Also, as with rotations, we can undo this operation using the Logarithm:

$$(\xi, \tau) = \ln(g)$$

Formulas for these expressions are given at the end of the notes.

An important use of these expressions though is the ability to describe the configuration of a robot manipulator between two different configurations. For example, consider the following example.

Example Use of Exponential Coordinates

Assume you have object that undergoes a transformation from g_i to g_f where,

$$g_i = \left[\begin{array}{ccc|c} \frac{1}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{7}}{3\sqrt{2}} & 1 \\ \frac{1}{3} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{7}}{3\sqrt{2}} & 3 \\ \frac{\sqrt{7}}{3} & 0 & -\frac{2}{3\sqrt{2}} & 2 \\ \hline & 0 & & 1 \end{array}\right] \quad \text{and} \quad g_f = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & \frac{4}{5} & -\frac{3}{5} & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} & -1 \\ \hline & 0 & & 1 \end{array}\right].$$

The Lie algebra element ξ with time τ can be computed using the Logarithm:

$$(\xi, \tau) = \ln\left(g_i^{-1} g_f\right)$$

which will take the object from g_i to g_f according to the product, $g_f = g_i \exp(\hat{\xi}\tau)$.

Then, the exponential can be used to average the two configurations in a natural way. Since $\exp(\hat{\xi}t)$ for $t \in [0, \tau]$ defines a trajectory from the identity element to $g = \exp(\hat{\xi}\tau)$, we can flow along the path to get a Lie group element between e and g. For example, we can solve for the half-way point between e and g, by solving for the exponential when $t = \tau/2$.

Explicitly, the first step of this process is to compute the tranformation from g_i to g_f:

$$\begin{aligned}
g_{if} &= g_i^{-1} g_f \\
&= \left[\begin{array}{c|c} R_i^\top & -R_i^\top d_i \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} R_f & d_f \\ \hline 0 & 1 \end{array}\right] \\
&= \begin{bmatrix} 0.3333 & 0.7958 & 0.5055 & -3.6458 \\ 0.7071 & -0.5657 & 0.4243 & 2.1213 \\ 0.6236 & 0.2160 & -0.7513 & -0.4566 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix}
\end{aligned}$$

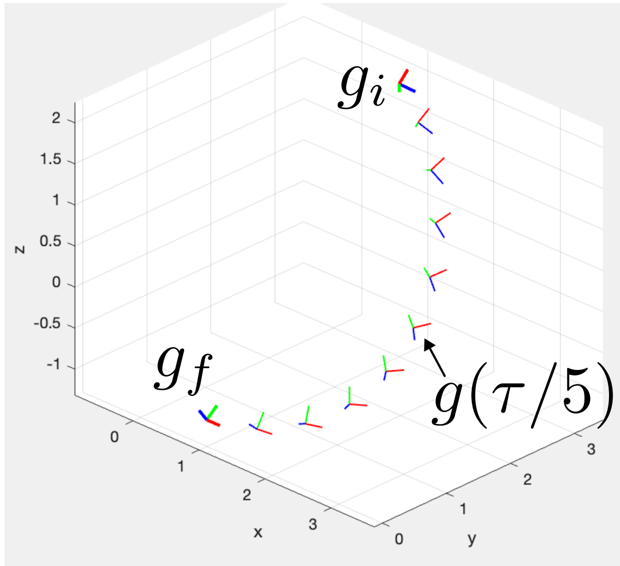
Then, we can compute the Lie-algebra element for the transformation g_{if} as (and choosing $\tau = 1$):

$$\begin{aligned}
\|\omega\| &= \frac{1}{\tau}\left(\cos^{-1}\left(\frac{\text{Trace}(R) - 1}{2}\right)\right) \\
&= \cos^{-1}\left(\frac{\text{Trace}(R) - 1)}{2}\right) \\
&= 3.0136 \\
[\omega]_\times &= \frac{\|\omega\|}{2\sin(\|\omega\|\tau)}(R - R^\top) \\
&= \begin{bmatrix} 0 & 1.0472 & -1.3939 \\ -1.0472 & 0 & 2.4581 \\ 1.3939 & -2.4581 & 0 \end{bmatrix} \\
\omega &= \begin{Bmatrix} -2.4581 \\ -1.3939 \\ -1.0472 \end{Bmatrix} \\
v &= \|\omega\|^2 \left((I - R)[\omega]_\times + \tau\omega\omega^\top\right)^{-1} d \\
&= \begin{Bmatrix} -3.3662 \\ -2.0419 \\ 4.4287 \end{Bmatrix}
\end{aligned}$$

Then, we can compute the any point in time $t \in [0, \tau]$ between g_i and g_f by solving for the exponential when $t = \tau/2$:

$$g(t) = \exp(\hat{\xi}t)g(0)$$

Visually, this gives us the following path:



Body and Spatial Velocity

The next topic we still need to cover is the concept of body and spatial velocity. In words, the body velocity is the velocity of the origin of the body coordinate frame relative to the spatial frame, as viewed in the current body frame. The spatial velocity is the velocity of a (possibly imaginary) point on the rigid body which is traveling through the origin of the spatial frame. The mathematical equations for these quantities are derived using the body and spatial twists:

$$\hat{\xi}_s = \dot{g}_{sb} g_{sb}^{-1}$$
$$\hat{\xi}_b = g_{sb}^{-1} \dot{g}_{sb}$$

Which give us the expressions:

$$\hat{\xi}_s = \begin{bmatrix} \dot{R}R^\top & -\dot{R}R^\top d + \dot{d} \\ 0 & 0 \end{bmatrix}$$
$$\hat{\xi}_b = \begin{bmatrix} R^\top \dot{R} & R^\top \dot{d} \\ 0 & 0 \end{bmatrix}$$

We can relate these two twists by the adjoint operation:

$$\begin{aligned} \hat{\xi}_s &= \mathrm{Ad}_{g_{sb}} \hat{\xi}_b \\ &= g_{sb} \hat{\xi}_b g_{sb}^{-1} \end{aligned}$$

The Adjoint Operation

In general, this adjoint operation is used to change the frame of reference. Our classic example is if we wanted to change some transformation that was in the end-effector frame to a transformation of the tool frame. Specifically, assume that we have the transformations g_{we} and $g_{we'}$. We can use these to compute the transformation that the end-effector undergoes by:

$$g_{ee'} = g_{we}^{-1} g_{we'}$$

If we also have the transformation between the end-effector and some tool in the end-effector's grip, g_{et}, then we can use the adjoint to obtain the transformation of the tool:

$$g_{tt'} = g_{et}^{-1} g_{ee'} g_{et} = \mathrm{Ad}_{g_{et}^{-1}} g_{ee'}$$

Always pay attention to the form of your adjoint operation. If you are given a different coordinate frame relationship (such as g_{te}), then your inverse will be in the opposite order:

$$g_{tt'} = g_{te} g_{ee'} (g_{te})^{-1} = \mathrm{Ad}_{g_{te}} g_{ee'}$$

Provided below are some useful equations. In $SE(2)$, the exponential and the logarithm are:

$$\exp(\hat{\xi}\tau) = \begin{cases} \left[\begin{array}{c|c} R(\xi^3\,\tau) & \frac{1}{\xi^3}(I - R(\xi^3\tau))\mathbb{J}\begin{Bmatrix}\xi^1\\ \xi^2\end{Bmatrix} \\ \hline 0 & 1 \end{array}\right] & \text{if } \xi^3 \neq 0 \\ \left[\begin{array}{c|c} I & \begin{Bmatrix}\xi^1\\ \xi^2\end{Bmatrix}\tau \\ \hline 0 & 1 \end{array}\right] & \text{if } \xi^3 = 0 \end{cases}$$

$$\ln_\tau(g) = \begin{cases} \omega = \frac{1}{\tau}\text{atan}(R_{21}, R_{11}) & \\ v = -\omega\mathbb{J}\,(I-R)^{-1}\,d & \text{if } \omega \neq 0\ , \\ v = \frac{1}{\tau}d & \text{if } \omega = 0 \end{cases} \quad \text{with } \xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}.$$

The skew-symmetric matrix $\mathbb{J}$ is:

$$\mathbb{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In $SO(3)$, barring the weird edge cases, the exponential and logarithm are:

$$\exp(\hat{\omega}\tau) = \begin{cases} I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\tau) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|\tau)) & \text{if } \omega \neq 0 \\ I & \text{if } \omega = 0. \end{cases}$$

$$\ln R = \begin{cases} \|\omega\| = \frac{1}{\tau}\cos^{-1}\left(\frac{\text{trace}(R)-1}{2}\right) & \\ \hat{\omega} = \frac{\|\omega\|}{2\sin(\|\omega\|\tau)}\left(R - R^T\right) & \text{if } \|\omega\| \neq 0\ \cdot \\ \hat{\omega} = 0 & \text{if } \|\omega\| = 0 \end{cases}$$

In $SE(3)$, the exponential and the logarithm are:

$$\exp(\hat{\xi}\tau) = \begin{cases} \left[\begin{array}{c|c} \exp(\hat{\omega}\,\tau) & (I - \exp(\hat{\omega}\tau))\frac{\hat{\omega}\,v}{\|\omega\|^2} + \frac{\omega\omega^T}{\|\omega\|^2}v\tau \\ \hline 0 & 1 \end{array}\right] & \text{if } \omega \neq 0 \\ \left[\begin{array}{c|c} I & v\tau \\ \hline 0 & 1 \end{array}\right] & \text{if } \omega = 0 \end{cases}$$

$$\ln(g) = \begin{cases} \omega = \ln_\tau(R) & \\ v = \|\omega\|^2\ \left(\left((I-R)\,\hat{\omega} + \tau\omega\omega^T\right)\right)^{-1} d & \text{if } \omega \neq 0\ \ , \\ v = \frac{1}{\tau}d & \text{if } \omega = 0. \end{cases} \quad \text{with } \xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}.$$

In these equations, the hat of $\omega \in \mathbb{R}^3$ is the skew-symmetric matrix:

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}.$$