Topics Covered:

- Introduction to Exponential Representation
- Exponential Representation of Rotations

Additional Reading:

- MLS Chapter 2, Section 2.2; LP 3.2.3

Exponential Representation of Group Motion / Displacements

In linear control, we often describe the dynamics of a linear system as:

$$\dot{x} = Ax$$

where x is our state variable of size n and A is a matrix of size $n \times n$. In the case of a linear time-invariant system, the solution to this differential equation is:

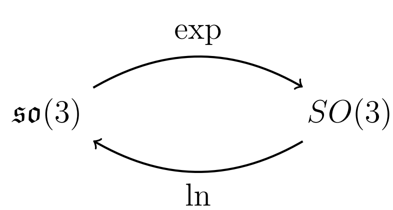
$$x(t) = e^{At}x(0)$$

where e^{At} is known as the matrix exponential. We define this matrix exponential as:

$$e^{At} = \sum_{n=1}^{\infty} \frac{(At)^n}{n!}$$

In this class, we are going to using this same Exponential Mapping to map between Lie Groups ($SO(3)$ and $SE(3)$) and their corresponding Lie Algebras ($\mathfrak{so}(3)$ and $\mathfrak{se}(3)$).

We will go over these expressions later, but in general you can think of the following map:

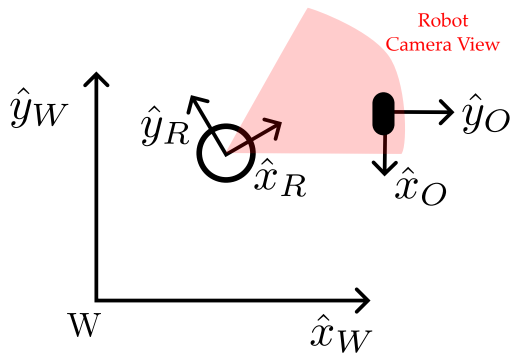


Here, $\exp$ is called the exponential map, and $\ln$ (the inverse operation) is called the Logarithm. These correspond to the matrix exponential and matrix logarithm operations respectively.

In general, $\exp$ and $\ln$ are useful for translating between finite transformations (rotations or rigid body motions) and their infinitesimal generators (like angular velocities and twists), allowing smooth and efficient computations in 3D space.

Example of Exponential Representation

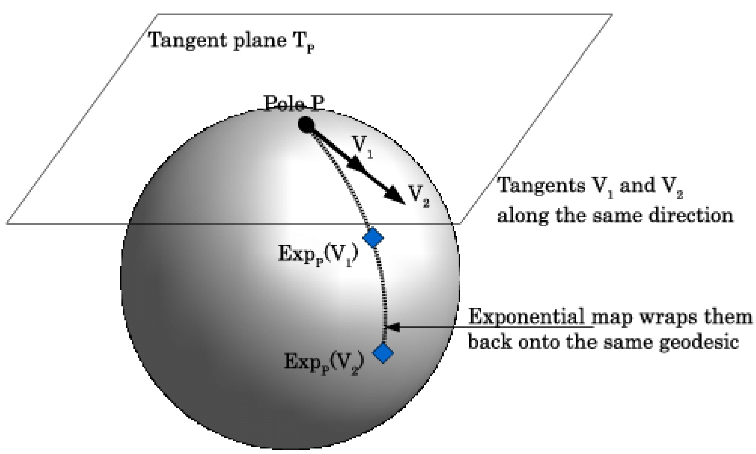
Let's consider the example you had from a previous homework where there is a mobile robot in the 3D world.



The mobile robot obtains measurements of it's orientation using an inertial measurement unit (IMU) that provides us with angular velocities. We can then compute the exponential map at each time step to obtain how the orientation of the robot changes over time. For example, if the angular velocity readings of the robot are $\omega(t) = [0, 0, 0.1]^\top$:

$$R(0 + \delta t) = \exp([\omega]_\times \delta t) R(0)$$

A visual representation of this mapping is the following (we will discuss in class):



One main benefit of using these exponential representations is that they allow us to smoothly interpolate between different orientations and to integrate rotational dynamics over time.

Method	Pros	Cons	When to Use
Exponential Map	Accurate for integrating rotations, natural for Lie groups	Requires matrix exponentiation, more complex mathematically	Advanced robotics and control applications
Quaternions	Compact, no gimbal lock, efficient for integration	Slightly more abstract than Euler angles	Real-time systems (e.g., drones, IMU fusion)
Euler Angles	Intuitive, simple for small rotations	Subject to gimbal lock, less efficient for complex rotations	Basic applications, small-angle motions

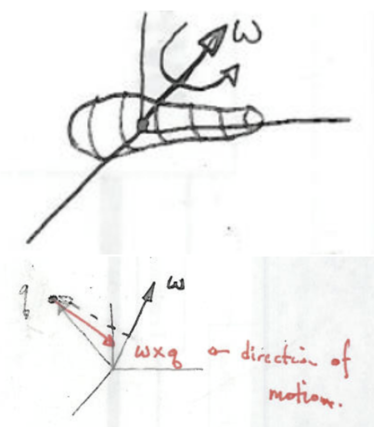
Table 1: Comparison of Rotation Representation Methods

We will cover $\exp$, $\ln$ for each group and their properties. In this lecture, we will cover the special orthogonal group, with the following lecture covering the remaining two. Outline:

1. $SO(3)$ and $\mathfrak{so}(3)$
2. $SE(3)$ and $\mathfrak{se}(3)$
3. $SE(2)$ and $\mathfrak{se}(2)$

Exponential Representation for $SO(3)$ and $\mathfrak{so}(3)$

In the context of rotations, $SO(3)$ represents the special orthogonal group in 3D space, which consists of 3x3 orthogonal matrices with determinant 1 that represent 3D rotations. The Lie algebra $\mathfrak{so}(3)$ represents the set of 3x3 skew-symmetric matrices, which correspond to angular velocities (or "infinitesimal rotations").



consider a point on a body undergoing rotation:

$$\dot{q}(t) = \omega \times q(t) = [\omega]_\times q(t) \quad (1)$$

here, ω is the axis of rotation, while $[\omega]_\times$ is the axis represented as a matrix

If we integrate this velocity, we get:

$$q(t) = e^{[\omega]_\times \tau} q(0)$$

WAIT: what is $e^{[\omega]_\times \tau}$? It is the solution to differential equation in (1).

From $\mathfrak{so}(3)$ to $SO(3)$ using the Exponential Map

The matrix exponential function $\exp : \mathfrak{so}(3) \to SO(3)$ maps a skew-symmetric matrix (which encodes a rotation axis and angle) to a rotation matrix. This is often combined with some "units of time", τ. This operation essentially "exponentiates" the infinitesimal rotation to get a finite rotation.

Mathematically:

$$R(\omega, \tau) = e^{[\omega]_\times \tau} \in SO(3)$$

Later we will derive the form of this computating via Rodrigues' rotation formula.

From $SO(3)$ to $\mathfrak{so}(3)$ using the Logarithm

The matrix logarithm function $\ln : SO(3) \to \mathfrak{so}(3)$ maps a rotation matrix back to a skew-symmetric matrix, which corresponds to the angular velocity or axis-angle representation of the rotation.

Mathematically:

$$[\omega]_\times = \ln(R) \in \mathfrak{so}(3)$$

Computing the Exponential Map for Rotations

The definition of the exponent of a matrix is:

$$e^{\mathrm{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{A}^n \qquad \text{(from Taylor series for } e^{\mathrm{A}}\text{)}$$

$$e^{[\omega]_\times \tau} = \sum_{n=0}^{\infty} \frac{1}{n!} ([\omega]_\times \tau)^n$$

If rotated for τ units of time, then:

$$R(\omega, \tau) = e^{[\omega]_\times \tau}$$

Can we compute $e^{[\omega]_\times \tau}$ without requiring the infinite series expansion definition? Yes! We will do this with the help of *Rodrigues' formula*. Most derivations of this formula assume that $[\omega]_\times$ has unit magnitude ($\|\omega\| = 1$). This allows us to observe the following. For simplicity of notation we will denote $[\omega]_\times$ as the matrix W.

$$W^2 = WW = -W^\top W = -I$$

Using this assumption, we can observe the following series:

$$\begin{aligned}
W^3 &= W^2W = -W \\
W^4 &= W^2W^2 = -W^2 \\
W^5 &= W^2W^3 = -W^3 = W \\
W^6 &= W^2W^4 = -W^4 = W^2 \\
W^7 &= W^2W^5 = -W \\
W^8 &= W^2W^6 = -W^2 \\
W^9 &= W^2W^7 = W \\
W^{10} &= W^2W^8 = W^2 \\
&\vdots
\end{aligned}$$

Ultimately, we should see a pattern emerge that allows us to simplify the series expansion:

$$\begin{aligned}
e^{[\omega]_\times \tau} &= \sum_{n=0}^{\infty} \frac{1}{n!}([\omega]_\times \tau)^n \\
&= I + [\omega]_\times \tau + [\omega]_\times^2 \frac{\tau^2}{2!} + [\omega]_\times^3 \frac{\tau^3}{3!} + \cdots \\
&= I + \left(\tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!}\right)[\omega]_\times + \left(\frac{\tau^2}{2!} - \frac{\tau^4}{4!} + \frac{\tau^6}{6!}\right)[\omega]_\times^2
\end{aligned}$$

Recalling the series expansions for $\sin(t)$ and $\cos(t)$:

$$\begin{aligned}
\sin(t) &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots \\
\cos(t) &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots
\end{aligned}$$

Therefore, our expression from before can be further simplified.

$$\begin{aligned}
e^{[\omega]_\times \tau} &= I + \underbrace{\left(\tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!}\right)}_{\sin(\tau)}[\omega]_\times + \underbrace{\left(\frac{\tau^2}{2!} - \frac{\tau^4}{4!} + \frac{\tau^6}{6!}\right)}_{1-\cos(\tau)}[\omega]_\times^2 \\
&= I + [\omega]_\times \sin(\tau) + [\omega]_\times^2 (1 - \cos(\tau))
\end{aligned}$$

This expression is Rodrigues' Formula!

Aside: without the assumption that $\|\omega\| = 1$, a similar relationship can be found using the following Lemma:

Lemma. Given $a \in \mathbb{R}^3$ such that $[a]_\times \in so(3)$, then:

$$[a]_\times^2 = aa^\top - \|a\|^2 I, \text{ and}$$
$$[a]_\times^3 = -\|a\|^2 [a]_\times$$

Then, plugging these relationships into our series expansion would yield the modified version of Rodrigues' Formula:

$$e^{[\omega]_\times \tau} = I + \frac{[\omega]_\times}{\|\omega\|} \sin(\|\omega\|\tau) + \frac{[\omega]_\times^2}{\|\omega\|^2}(1 - \cos(\|\omega\|\tau))$$

Next, can we show that $e^{\omega\tau} \in SO(3)$ (i.e., that $e^{\omega\tau}$ produces a valid rotation matrix)?

Well, $R \in SO(3) \implies R^\top R = I$ and $\det(R) = 1$

a) $(e^{[\omega]_\times \tau})^\top (e^{[\omega]_\times \tau}) = e^{[\omega]_\times^\top \tau} e^{[\omega]_\times \tau} = e^{-[\omega]_\times \tau} e^{[\omega]_\times \tau} = I$

b) $\det(e^{[\omega]_\times \tau}) = e^{\mathrm{Tr}[\omega]_\times \tau} = e^0 = 1$

Thus, $e^{[\omega]_\times \tau} \in SO(3)$ is true. But what about the other way around?

The exponential map is surjective onto $SO(3)$ (Proposition 2.5 from MLS):

Proposition 1. *Given $R \in SO(3)$, there exists an $\omega \in \mathbb{R}^3$ where $\|\omega\| = 1$, and a $\tau \in \mathbb{R}$ such that $R = \exp([\omega]_\times \tau)$.*

Note that this is like saying that there exists a function taking in an R and returning a pair (ω, τ) such that $R = \exp(\omega\tau)$ where $\|\omega\| = 1$.
This function will be called the logarithm and will be denoted by $\ln$,

$$(\omega, \tau) = \ln(R), \quad \text{where } \|\omega\| = 1.$$

If we allow for $\|\omega\| = \tau$, then we will write

$$\omega = \ln R$$

The pair (ω, τ) are not necessarily unique!

Proof of Proposition:

Proof. The proof is a little nasty and constructive.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Note: only 3 of these entries are uniquely determined, (ω, τ) has only 3 unique variables too.

$$e^{[\omega]_\times \tau} = I + [\omega]_\times \sin(\tau) + [\omega]_\times^2 (1 - \cos(\tau))$$

$\implies$ let $s_\tau = \sin(\tau), c_\tau = \cos(\tau), v_\tau = (1 - \cos(\tau))$

$$= \begin{bmatrix} 1 - v_\tau((\omega^2)^2 + (\omega^3)^2) & \omega^1\omega^2 v_\tau - \omega^3 s_\tau & \omega^1\omega^3 v_\tau + \omega^2 s_\tau \\ \omega^1\omega^2 v_\tau + \omega^3 s_\tau & 1 - v_\tau((\omega^1)^2 + (\omega^3)^2) & \omega^2\omega^3 v_\tau - \omega^1 s_\tau \\ \omega^1\omega^3 v_\tau - \omega^2 s_\tau & \omega^2\omega^3 v_\tau + \omega s_\tau & 1 - v_\tau((\omega')^2 + (\omega^2)^2) \end{bmatrix}$$

$$= \begin{bmatrix} (\omega^1)^2 v_\tau + c_\tau & \omega^1\omega^2 v_\tau - \omega^3 s_\tau & \omega^1\omega^3 v_\tau + \omega^2 s_\tau \\ \omega^1\omega^2 v_\tau + \omega^3 s_\tau & (\omega^2)^2 v_\tau + c_\tau & \omega^2\omega^3 v_\tau - \omega^1 s_\tau \\ \omega^1\omega^3 v_\tau - \omega^2 s_\tau & \omega^2\omega^3 v_\tau + \omega^1 s_\tau & (\omega^3)^2 v_\tau + c_\tau \end{bmatrix}$$

Equating matrix elements should give the answer. It will suffice to equate enough coefficients to find ω and τ.

$a)$ examine the trace.

$$\begin{aligned} \mathrm{Tr}(R) &= r_{11} + r_{22} + r_{33} \\ \mathrm{Tr}(e^{[\omega]_\times \tau}) &= 1 + 2\cos(\tau) \\ \implies \quad 1 + 2\cos(\tau) &= \mathrm{Tr}(R) \\ \implies \quad \cos(\tau) &= \frac{\mathrm{Tr}(R) - 1}{2} \\ \implies \quad \tau &= \cos^{-1}\left(\frac{\mathrm{Tr}(R) - 1}{2}\right) \end{aligned}$$

This solution can be zero, and can go to $\pm 2\pi n$ and still give the same answer.

If $\tau = 0$, then ω can be anything.

If $\tau \neq 0$, then need to find ω.

$b)$ examine R.

Well, if the look at the off-diagonal terms of $R - R^\top$ and $e^{[\omega]_\times \tau} - (e^{[\omega]_\times \tau})^\top = e^{-[\omega]_\times \tau} - e^{-[\omega]_\times \tau}$, we have

$$\begin{aligned} r_{32} - r_{23} &= 2\omega^1 \sin(\tau) \\ r_{13} - r_{31} &= 2\omega^2 \sin(\tau) \\ r_{21} - r_{12} &= 2\omega^3 \sin(\tau) \end{aligned}$$

$\Longrightarrow$

$$\omega = \frac{1}{2\sin\tau}\begin{Bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{Bmatrix}$$

□

So, now that we got the answer, does it make sense?
Well, first off,

$$\mathrm{Tr}(R) = \Sigma\lambda_i$$

or the sum of eigenvalues, what are these like? $det(R) = \prod \lambda_i = 1$ or $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1$. One eigenvalue is $\lambda_1 = 1$, which leaves $\lambda_2 \cdot \lambda_3 = 1$. The others are two complex conjugates $\lambda \cdot \lambda^\star = 1$

$$\begin{aligned} \mathrm{Tr}(R) &= 1 + \lambda + \lambda^\star \\ &= 1 + 2\mathrm{Re}(\lambda) \qquad \text{(between 0 and 1)} \end{aligned}$$

$\Longrightarrow$

$$1 + 2\mathrm{Re}(\lambda) = 1 + 2\cos(\tau)$$

Therefore, a solution is possible.

In summary:

Given $\omega \in \mathbb{R}^3$ such that $[\omega]_\times \in so(3)$ (and $\|\omega\| = 1$),

$$e^{[\omega]_\times \tau} = I + [\omega]_\times \sin(\tau) + [\omega]_\times^2(1 - \cos(\tau)) \in SO(3)$$

Given $R \in SO(3)$, there exists ($\exists$) $\omega \in \mathbb{R}^3$, $\tau \in \mathbb{R}$ where $\|\omega\| = 1$, defined by

$$\tau = \cos^{-1}\left(\frac{\mathrm{Tr}(R) - 1}{2}\right)\begin{cases} \tau = 0, & \omega \text{ is arbitrary} \\ \tau \neq 0, & \omega = \frac{1}{2\sin(\theta)}\begin{Bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{Bmatrix} \end{cases}$$

and is written as

$$(\omega, \tau) = \ln(R)$$

or if $\|\omega\|$ can be non-unit

$$\omega = \ln R$$

In this case $\|\omega\|$ encodes for the amount of rotation as did the τ. Then:

$$\|\omega\| = \cos^{-1}\left(\frac{\mathrm{Tr}(R) - 1}{2}\right)$$
$$\frac{\omega}{\|\omega\|} = \frac{1}{2\sin(\|\omega\|)}\begin{Bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{Bmatrix}$$