Topics Covered:

- Subscript Cancellation Rule
- Angular Velocity
- Twists
- Example

Additional Reading:

- LP 3.2.2 (Angular Velocity), 3.3.2 (Twists)
- MLS Chapter 2, Section 4

Review

Last week we introduced the Lie algebra element $\mathfrak{se}(2)$ and $\mathfrak{se}(3)$ as the tangent space at the identity of $SE(2)$ and $SE(3)$, respectively. We also showed that this element is derived as the time derivative of a transformation matrix $g(t)$ at $t = 0$:

$$\hat{\xi} = \frac{d}{dt} g(t)\bigg|_{t=0} = \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)/\mathfrak{se}(3)$$

where $\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix} \in \mathbb{R}^3/\mathbb{R}^6$ is called a **twist**.

In this lecture, we will discuss the difference between spatial and body twists.

Definition: Subscript Cancellation Rule

Page 62 of Modern Robotics by Kevin Lynch and Frank Park (LP) describes the "Subscript Cancellation Rule" as follows: When multiplying two rotation matrices, if the second subscript of the first matrix matches the first subscript of the second matrix, the two subscripts "cancel" and a change of reference frame is achieved:

$$R_{ab} R_{bc} = R_{a\not{b}} R_{\not{b}c} = R_{ac}$$

A rotation matrix is just a collection of three unit vectors, so the reference frame of a vector can also be changed by a rotation matrix using a modified version of the Subscript Cancellation Rule:

$$R_{ab} p_b = R_{a\not{b}} p_{\not{b}} = p_a$$

The subscript cancellation rule also extends to transformations by considering some arbitrary reference frames a, b and c and some vector v expressed in frame b:

$$g_{ab}g_{bc} = g_{a\not{b}}g_{\not{b}c} = g_{ac}$$

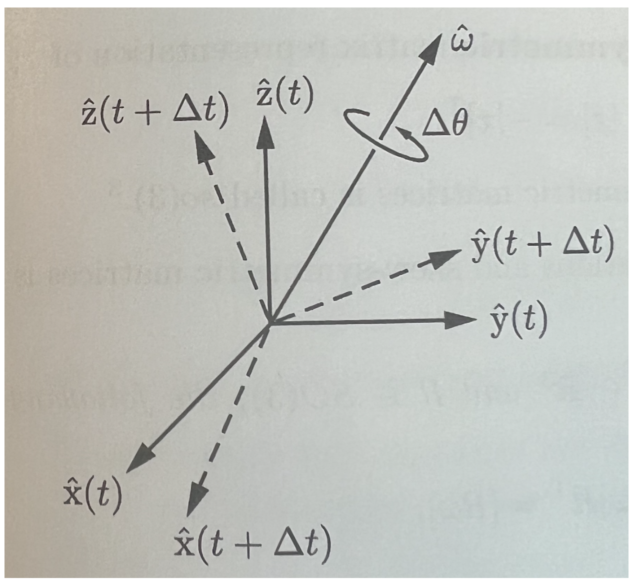
and

$$g_{ab}v_b = g_{a\not{b}}v_{\not{b}} = v_a$$

where v_a is the vector v expressed in frame a.

Angular Velocity

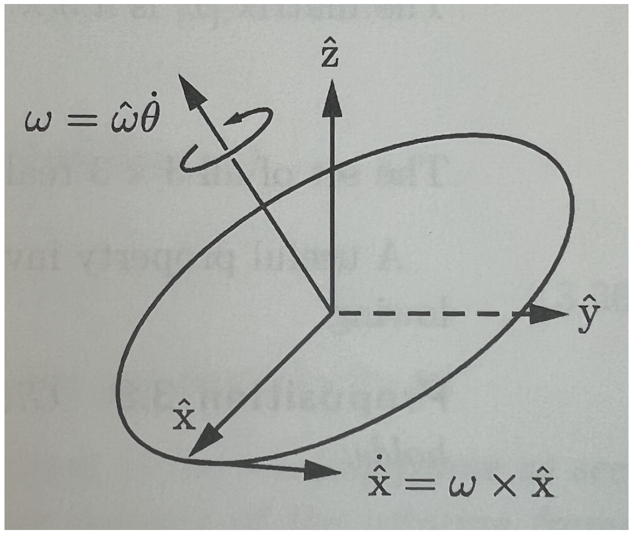
Consider the following frame attached to a rotating body:



If we examine the body frame at times t and $t+\delta t$, the change in frame orientation can be described as a rotation of angle $\delta\theta$ about some unit axis $\hat{\omega}$ passing through the origin. As the limit $\delta t \to 0$, we can define the angular velocity ω as:

$$\omega = \hat{w}\dot{\theta}$$

This angular velocity is illustrated as follows:



From the figure, we can decompose the individual coordinate axis velocities as:

$$\dot{\hat{x}} = \omega \times \hat{x}$$
$$\dot{\hat{y}} = \omega \times \hat{y}$$
$$\dot{\hat{z}} = \omega \times \hat{z}$$

To express these equations in coordinates, we must choose a reference frame for ω. Two natural choices are the fixed frame s and the body frame b. This will later be discussed as resulting in either *spacial velocity* or *body velocity*.

Starting with spacial frame, let $\omega_s \in \mathbb{R}^3$ be the angular velocity expressed in fixed-frame coordinates. Additionally let $R(t)$ be the rotation matrix describing the orientation of the body frame with respect to the fixed frame at time t (i.e., $R_{sb}(\theta(t))$). Each column of R then denotes a coordinate frame axis in fixed-frame coordinates, denotes as

$$R = \begin{bmatrix} | & | & | \\ r_1 & r_2 & r_3 \\ | & | & | \end{bmatrix}$$

Thus, the time rate of change for R can be expressed as:

$$\dot{R} = \begin{bmatrix} \omega_s \times r_1 & \omega_s \times r_2 & \omega_s \times r_3 \end{bmatrix} = \omega_s \times R$$

Aside into cross products:

Given two vectors a and $b \in \mathbb{R}^3$, the cross product $a \times b$ represents the vector that is orthogonal (perpendicular) to both a and b with the direction determined by the right-hand rule.

This product is defined as the determinant of:

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where i, j, and k are unit vectors in x, y, and z directions, respectively.
This is equivalent to the expression:

$$a \times b = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

We're going to use our trick with skew-symmetric matrices to get rid of the cross product.

Skew symmetric matrices:

Given a vector $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top \in \mathbb{R}^3$, then the skew-symmetric matrix for x is

$$[x]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Thus, by converting ω_s into a skew-symmetric matrix, we can eliminate the cross product as:

$$\begin{aligned} \dot{R} &= \omega_s \times R \\ &= [\omega_s]_\times R \end{aligned}$$

Skew symmetric matrices for planar rotations:

Note, in 2D our element ω is one-dimensional, so our "cross product" is equivalent to rotating a vector in the plane. This rotation can be conceptualized as a *perpendicular operator* represented by the skew-symmetric matrix:

$$[\omega]_\times = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

Thus, for 2D, we denote $\omega \times R$ as:

$$\omega \times R = [\omega]_\times R$$

Continuing with our previous expression for the time rate of change of R, we can express the angular velocity in fixed-frame coordinates by post-multiplying both sides by R^{-1} gives us the expresson for the spacial angular velocity:

$$\begin{aligned}\dot{R} &= \omega_s \times R \\ &= [\omega_s]_\times R \\ [\omega_s]_\times &= \dot{R}R^{-1}\end{aligned}$$

Now, if we consider ω_b to be expressed in body-frame coordinates, we can solve for the body frame velocity using the transformation:

$$\omega_s = R_{sb}\omega_b$$

Rearranging this expression also gives us the expression for the body angular velocity:

$$\omega_b = R_{sb}^{-1}\omega_s = R^{-1}\omega_s = R^\top \omega_s$$

Expressing the body-frame angular velocity in skew-symmetric matrix form yields:

$$\begin{aligned}[\omega_b]_\times &= [R^\top \omega_s]_\times \\ &= R^\top [\omega_s]_\times R \qquad \text{(proof shown below)} \\ &= R^\top (\dot{R}R^\top)R \\ &= R^\top \dot{R} = R^{-1}\dot{R}\end{aligned}$$

Property of skew-symmetric matrices:

(From Proposition 3.8 of LP)
Given any $\omega \in \mathbb{R}^3$, and $R \in SO(3)$, the following always holds:

$$R^\top [\omega]_\times R = [R^\top \omega]_\times$$

Proof. Letting $r_i^\top$ be the ith row of R, we have:

$$\begin{aligned}R^\top [\omega]_\times R &= \begin{bmatrix} r_1^\top \\ r_2^\top \\ r_3^\top \end{bmatrix} \begin{bmatrix} \omega \times r_1 & \omega \times r_2 & \omega \times r_3 \end{bmatrix} \\ &= \begin{bmatrix} r_1^\top(\omega \times r_1) & r_1^\top(\omega \times r_2) & r_1^\top(\omega \times r_3) \\ r_2^\top(\omega \times r_1) & r_2^\top(\omega \times r_2) & r_2^\top(\omega \times r_3) \\ r_3^\top(\omega \times r_1) & r_3^\top(\omega \times r_2) & r_3^\top(\omega \times r_3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -r_3^\top \omega & r_2^\top \omega \\ r_3^\top \omega & 0 & -r_1^\top \omega \\ -r_2^\top \omega & r_1^\top \omega & 0 \end{bmatrix} \\ &= [R^\top \omega]_\times\end{aligned}$$

Definition: Angular Velocity in Fixed-Frame and Body-Frame (Prop 3.9 of LP)

Let $R(t)$ denote the orientation of the rotating frame as seen from the fixed frame. Denote ω as the angular velocity of the rotating frame. Then:

$$[\omega_s]_\times = \dot{R}R^{-1}$$
$$[\omega_b]_\times = R^{-1}\dot{R} = R^\top \dot{R}$$

Twists

These formulas can be used to derive two more formulas for body and spatial twist in homogeneous coordinates.

body twist:

$$\begin{aligned}\hat{\xi}_b = g_{sb}^{-1}\dot{g}_{sb} = g_{b\not{s}}\dot{g}_{\not{s}b} &= \begin{bmatrix} R^\top & -R^\top d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{R} & \dot{d} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R^\top \dot{R} & R^\top \dot{d} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} [\omega_b]_\times & v_b \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)/\mathfrak{se}(3)\end{aligned}$$

spatial twist:

$$\begin{aligned}\hat{\xi}_s = \dot{g}_{sb}g_{sb}^{-1} = \dot{g}_{s\not{b}}g_{\not{b}s} &= \begin{bmatrix} \dot{R} & \dot{d} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R^\top & -R^\top d \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{R}R^\top & -\dot{R}R^\top d + \dot{d} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} [\omega_s]_\times & v_s \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)/\mathfrak{se}(3)\end{aligned}$$

Note that in both cases, we can get the vector form by "unhatting":

$$\xi_b = \left(\hat{\xi}_b\right)^\vee = \begin{Bmatrix} v_b \\ \omega_b \end{Bmatrix}, \qquad \xi_s = \left(\hat{\xi}_s\right)^\vee = \begin{Bmatrix} v_s \\ \omega_s \end{Bmatrix}$$

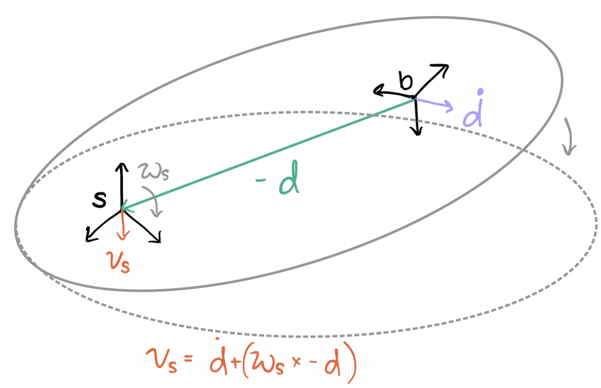
Also, note that in this formula, $v_s = -\dot{R}R^\top d + \dot{d}$ should NOT be interpreted as the linear velocity of the base-frame origin expressed in the fixed frame (that would simply be $\dot{d}$). Instead, the physical meaning is the instantaneous velocity of the body, currently at the fixed-frame origin, expressed in the fixed frame:

- ω_b: the angular velocity expressed in $\{b\}$
- ω_s: the angular velocity expressed in $\{s\}$
- v_b: the linear velocity of a point at the origin of $\{b\}$ expressed in $\{b\}$
- v_s: the linear velocity of a point at the origin of $\{s\}$ expressed in $\{s\}$

The intuition behind the expression for v_s can be derived as:

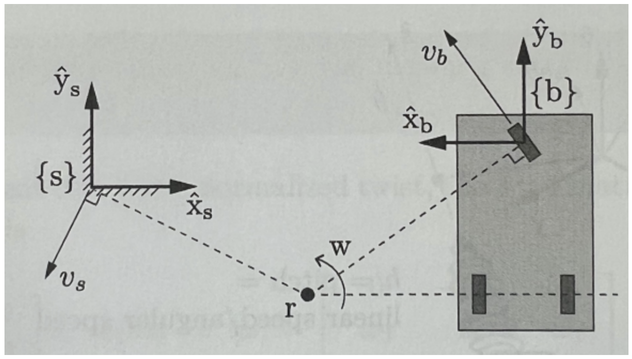
$$\begin{aligned} v_s &= \dot{d}_{sb} - \dot{R}R^\top d \\ &= \dot{d}_{sb} - [\omega_s]_\times d \\ &= \dot{d}_{sb} - \omega_s \times d \\ &= \dot{d}_{sb} + \omega_s \times -d \end{aligned}$$

This is summarized by the diagram:



Example

Consider the following example:



This example shows the top view of a car, with a single steerable front wheel driving on a plane. The angular velocity caused the the rotation of the front wheel is $\mathbf{w} = 2$ rad/s about point r. We can write the point of rotation in reference to either s or b as:

$$r_s = (2, -1, 0), \quad r_b = (2, -1.4, 0)$$

The angular velocity can then be expressed in either frame as:

$$\omega_s = (0, 0, 2), \quad \omega_b = (0, 0, -2)$$

From the figure, we can solve for spatial and body velocity of the car's frame as:

$$v_s = \omega_s \times (-r_s) = r_s \times \omega_s = (-2, -4, 0)$$
$$v_b = \omega_b \times (-r_b) = r_b \times \omega_b = (2.8, 4, 0)$$

Putting these together we can obtain the spatial and body twists as:

$$\xi_s = \begin{Bmatrix} -2 \\ -4 \\ 2 \end{Bmatrix}, \quad \xi_b = \begin{Bmatrix} 2.8 \\ 4 \\ -2 \end{Bmatrix}$$

Notice that we can also do this in 2D notation using $[\omega]_\times = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$:

$$v_s = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$
$$v_b = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1.4 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 4 \end{bmatrix}$$

Next class we will go over how to apply change of frame transformations to twists using the adjoint operation, i.e., $\hat{\xi}_s = \mathrm{Ad}_{g_{sb}} \hat{\xi}_b$.