Topics Covered:

- Introduction to Velocity
- Lie Algebras

Additional Reading:

- LP 3.2.2
- MLS Chapter 2, Section 3.2 and Section 4.1

The Space of Rigid Body Motions

Recall that space of planar rigid body motions is described by the Special Euclidean group $SE(2)$

One representation is homogeneous coordinates:

$$g = \begin{bmatrix} R & \vec{d} \\ 0 & 1 \end{bmatrix}, \quad d \in \mathbb{R}^{2\times 1} = E^2, \quad R \in SO(2)$$

The composition of $SE(2)$ is then written as $E^2 \times SO(2)$.

What is $SE(3)$, well it's the space of rigid body motions in 3D composed by $E^3 \times SO(3)$.

Here, $SO(3)$ is the special orthogonal group in 3D, which is the set of all 3×3 rotation matrices that are orthogonal and have determinant 1:

$$R \in SO(3) : \begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix}, \quad e_3 = e_1 \times e_2, \quad R^\top R = I, \quad det(R) = +1$$

Thus, in 3D our homogeneous coordinates become:

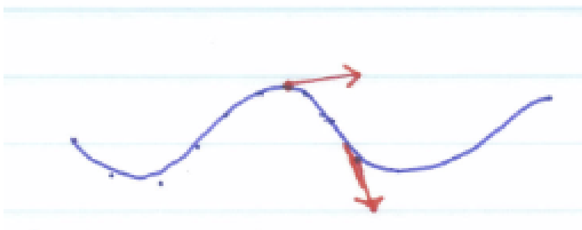
$$g = \begin{bmatrix} R & \vec{d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3\times 3 & 3\times 1 \\ 1\times 3 & 1\times 1 \end{bmatrix}$$

Now we want to ask: how can we describe rigid body *motion* rather than just rigid body transformations.

Introduction to Velocity

To think about motion, we need to go back to first principles.

- vectors arise from infinitesimal displacements associated to trajectories



curve $p(t) \Rightarrow$ time derivative $\dot{p}(t)$ is a vector (varies according to the curve)

What is a vector?

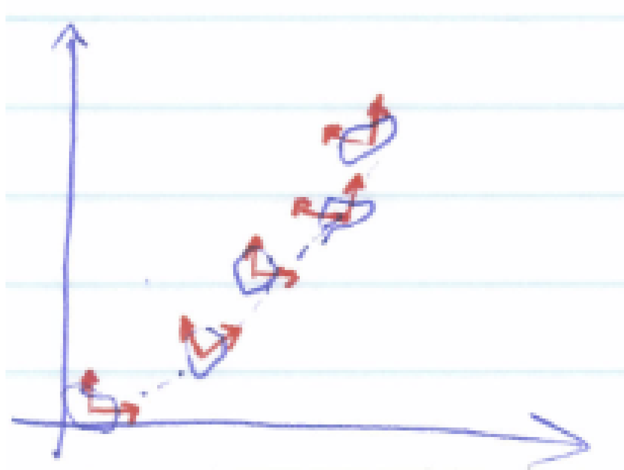
1. displacement/change between two Euclidean points $v = q_2 - q_1$
2. mathematical description of the incremental change when following a trajectory in Euclidean space

Properties of vectors:

1. form a linear space: (can add vectors & multiply by scalar)
2. transform under reference frame change
3. there is a mapping from displacements to vectors (and vice-versa)

Velocities in SE(2): So, let's start with a trajectory in SE(2), given by $g(t)$.

$$g(t) : [t_0, t_1] \to SE(2)$$



Question: What is the velocity of the body moving according to $g(t)$?

$$
\begin{aligned}
g(t) &= \begin{bmatrix} R(\theta(t)) & d(t) \\ 0 & 1 \end{bmatrix} \\
\dot{g}(t) &= \begin{bmatrix} \frac{d}{dt}R(\theta(t)) & \frac{d}{dt}d(t) \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

where the derivative of our rotation matrix can be computed as:

$$
\begin{aligned}
\frac{d}{dt}R(\theta(t)) &= \frac{d}{dt}\begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix} \\
&= \begin{bmatrix} -\sin(\theta)\dot{\theta} & -\cos(\theta)\dot{\theta} \\ \cos(\theta)\dot{\theta} & -\sin(\theta)\dot{\theta} \end{bmatrix} \\
&= \frac{\partial R(\theta)}{\partial \theta}\dot{\theta} \\
&= DR(\theta) \cdot \dot{\theta}
\end{aligned}
$$

Aside:

A quick note about notation. There are many ways to denote derivatives and partial derivatives, depending on your application. Consider the following for $f = x^2y$ and $g = \begin{bmatrix} x^2y \\ y^3 \end{bmatrix}$

Notation	Function Type	Name	Example
$\frac{\partial f}{\partial x}$	Scalar or vector	Partial derivative	$\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial g}{\partial x} = \begin{bmatrix} 2xy \\ 0 \end{bmatrix}$
∇f	Scalar	Gradient	$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \end{bmatrix}$
J	Vector	Jacobian	$J = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \\ 0 & 3y^2 \end{bmatrix}$

Since we want a shorthand notation of writing $\frac{\partial R}{\partial \theta}$, we will abuse notation and use D to denote this partial derivative. However, note that D is typically used to denote the total derivative of a scalar or vector-valued function.

Continuing with our notation, we can write our velocity of the body as:

$$
\dot{g}(t) = \begin{bmatrix} DR(\theta) \cdot \dot{\theta} & \dot{d} \\ 0 & 0 \end{bmatrix}
$$

To describe the change in velocity in the body frame, we can use a coordinate frame transformation:

$$\begin{aligned}
\dot{g}_B^B(t) &= g_A^B(t)\dot{g}_B^A(t) \\
&= \left(g_B^A(t)\right)^{-1}\dot{g}_B^A(t) \\
&= \begin{bmatrix} R^\top & -R^\top d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} DR(\theta)\cdot\dot{\theta} & \dot{d} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} R^\top DR(\theta)\dot{\theta} & R^\top \dot{d} \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Solving for the rotational component:

$$\begin{aligned}
R^\top DR(\theta)\cdot\dot{\theta} &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^\top \begin{bmatrix} -\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{bmatrix}\cdot\dot{\theta} \\
&= \begin{bmatrix} -\cos(\theta)\sin(\theta)+\sin(\theta)\cos(\theta) & -\cos(\theta)^2-\sin(\theta)^2 \\ \cos(\theta)^2+\sin(\theta)^2 & -\sin(\theta)\cos(\theta)+\cos(\theta)\sin(\theta) \end{bmatrix}\cdot\dot{\theta} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\cdot\dot{\theta} \\
&= [\dot{\theta}]_\times
\end{aligned}$$

This is the skew-symmetric matrix-version of the velocities. Note: a skew-symmetric matrix is defined as the following:

Definition: Skew-Symmetric Matrix

Given a vector $v = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$ the skew-symmetric matrix is defined as:

$$[v]_\times = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

One use of these matrices is that they transform cross products:

$$a \times b = [a]_\times b$$

Additional properties of skew-symmetric matrices include:

1. If A is skew-symmetric, then $A^\top = -A$
2. If A is skew-symmetric, then $a_{ji} = -a_{ij}$

Continuing with our derivation:

$$\dot{g}_B^B(t) = \begin{bmatrix} R^\top DR(\theta)\dot{\theta} & R^\top \dot{d} \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} [\dot{\theta}(t)]_\times & R^{-1}(\theta(t))\dot{d}(t) \\ 0 & 0 \end{bmatrix} \qquad \text{(where implicitly } \dot{d}(t) = \dot{d}_B^A(t)\text{)}$$

We can think of this as:

$$\dot{g}_B^B(t) = \begin{bmatrix} [\dot{\theta}(t)]_\times & \dot{d}_B^B(t) \\ 0 & 0 \end{bmatrix} \qquad \text{(This is called body velocity!)}$$

Because $\dot{g}_B^B(t)$ depends on three variables only, it is often convenient to write it that way. Furthermore, there is another symbol for velocities of SE(2) elements; it is ξ. (the symbol xi).

The velocity ξ is written as:

$$\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix} = \begin{Bmatrix} v_x \\ v_y \\ \omega \end{Bmatrix} \qquad (\xi \in \mathbb{R}^3 \text{ for SE(2)}, \xi \in \mathbb{R}^6 \text{ for SE(3)})$$

this can be thought of as the vector form of velocity for the vector form coordinates $g = \{x, y, \theta\}$.

Ok, but what is ξ in homogeneous representation?

$$\hat{\xi} = \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix}$$

(If you want to keep the notation separate from unit vectors, we can denote this as $(\cdot)^\wedge$)

Thus, from vector form of a vector to homogeneous form is called "hatting". The other way is called "unhatting" and will be denoted as $(\cdot)^\vee$.

The benefit of homogeneous coordinates will be that it will allow us to easily change between spacial and body velocity

$$\dot{g}_B^B = g_A^B \cdot \dot{g}_B^A$$
$$\Updownarrow$$
$$\hat{\xi}^B = g_A^B \cdot \hat{\xi}^A$$

Notably this matrix ξ is a Lie algebra element of our Lie group SE(2), i.e., $\hat{\xi} \in \mathfrak{se}(2)$. Similarly, $[\omega]_\times \in \mathfrak{so}(2)$. Here, the gothic font is used to denote the Lie algebra element of the Lie group. In short, the Lie algebra is the tangent space of the Lie group near the identity.

- For SE(2), the Lie algebra element is denoted $\mathfrak{se}(2)$

- For SE(3), the Lie algebra element is denoted $\mathfrak{se}(3)$
- For SO(2), the Lie algebra is denoted $\mathfrak{so}(2)$
- For SO(3), the Lie algebra is denoted $\mathfrak{so}(3)$

Definition: Lie algebra

A Lie algebra is a vector space $\mathfrak{g}$ together with an operation (called the Lie bracket) denoted $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity ($x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0$).

Notably, every Lie group gives rise to a Lie algebra, which is the tangent space at the identity.

Example:

- for $\mathfrak{so}(2)/\mathfrak{so}(3)$, the Lie algebra is the set of all possible $\dot{R}$ when $R = I$
- for $\mathfrak{se}(2)/\mathfrak{se}(3)$, the Lie algebra is the set of all possible $\dot{g}$ when $g = I$

To demonstrate the property of linearity for $\mathfrak{se}(2)$, consider the following:

$$\begin{aligned}
\xi_1 + \xi_2 &= \begin{Bmatrix} v_1 \\ \omega_1 \end{Bmatrix} + \begin{Bmatrix} v_2 \\ \omega_2 \end{Bmatrix} = \begin{Bmatrix} v_1 + v_2 \\ \omega_1 + \omega_2 \end{Bmatrix} \\
\hat{\xi}_1 + \hat{\xi}_2 &= \begin{bmatrix} [\omega_1]_\times & v_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} [\omega_2]_\times & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\omega_1]_\times + [\omega_2]_\times & v_1 + v_2 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} [\omega_1 + \omega_2]_\times & v_1 + v_2 \\ 0 & 0 \end{bmatrix} \\
&= (\xi_1 + \xi_2)^\wedge
\end{aligned}$$