Topics Covered:

- Introduction to Forces
- Wrenches
- Relation to Jacobians

Additional Reading:

- MLS Chapter 2, Section 5.1; Chapter 3, Section 4.2
- LP 3.4, 5.2

Introduction to Forces

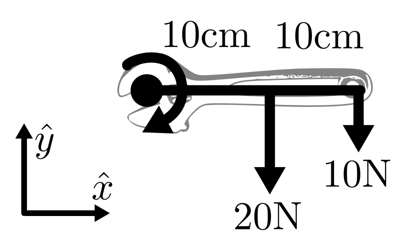
The traditional equation that we use to think about how force and torque are related is:

$$\tau = r \times F$$

with $\tau \in \mathbb{R}^3$ being a torque resulting from a force $F \in \mathbb{R}^3$ being applied at a point some distance $r \in \mathbb{R}^3$ away from the point where the torque is being measured. The key here, is that r and F must be represented in the same frame.

Note that the direction of this torque will obey our right-hand rule.

Consider the following example:



This example is the classic exampled used to demonstrate the concept of converting force to torque, because most people have an intuitive understanding of how a wrench works. Bascially, there are two equivalent ways to generate a torque:

$$\begin{aligned}\tau_1 &= r_1 \times F_1 \\ &= \begin{bmatrix}10\\0\\0\end{bmatrix} \times \begin{bmatrix}0\\-20\\0\end{bmatrix} \\ &= \begin{bmatrix}0\\0\\-200\end{bmatrix} Ncm = \begin{bmatrix}0\\0\\-2\end{bmatrix} Nm\end{aligned}$$

or

$$\begin{aligned}\tau_2 &= r_2 \times F_2 \\ &= \begin{bmatrix}0\\20\\0\end{bmatrix} \times \begin{bmatrix}10\\0\\0\end{bmatrix} \\ &= \begin{bmatrix}0\\0\\-200\end{bmatrix} Ncm = \begin{bmatrix}0\\0\\-2\end{bmatrix} Nm\end{aligned}$$

Wrenches

Similar to twists, we can merge the torque (moment) and the force into a single six-dimensional vector called a *wrench*. This is also sometimes called a *spatial force*:

$$F = \begin{bmatrix}f\\\tau\end{bmatrix} \in \mathbb{R}^6$$

This is similar to our notion of a twist, where $f \in \mathbb{R}^3$ is our linear component and $\tau \in \mathbb{R}^3$ is our rotational component.

This representation will allow us to change the frame of reference for an applied wrench using the adjoint transformation:

$$\begin{aligned}F_C &= \mathrm{Ad}^{\top}_{g_{BC}} F_B \\ &= \begin{bmatrix}R_{BC} & [d_{BC}]_\times R_{BC}\\ 0 & R_{BC}\end{bmatrix}^{\top} \begin{bmatrix}f_B\\\tau_B\end{bmatrix} \qquad \text{(Assuming } F = \begin{bmatrix}f\\\tau\end{bmatrix}\text{)} \\ &= \begin{bmatrix}R^{\top}_{BC} & 0\\ -R^{\top}_{BC}[d_{BC}]_\times & R^{\top}_{BC}\end{bmatrix} F_B \\ \begin{bmatrix}f_C\\\tau_C\end{bmatrix} &= \begin{bmatrix}R^{\top}_{BC} f_B\\ R^{\top}_{BC}\tau_B - R^{\top}_{BC}[d_{BC}]_\times f_B\end{bmatrix}\end{aligned}$$

This provides us with the individual transformations:

$$f_C = R_{BC}^{\top} f_B, \quad \tau_C = R_{BC}^{\top} \left(\tau_B - (d_{BC} \times f_B)\right)$$

Overall, we can see that the adjoint transforms the force and torque vectors from the B frame into the C frame and adds an additional torque of $-d_{BC} \times f_B$ to the torque vector.

Equivalent Wrenches

Let's derive why this adjoint transformation works. By definition:

two wrenches are *equivalent*
if they generate the same work for every possible rigid body motion.

The purpose of equivalent wrenches is that it allows us to rewrite a known wrench in terms of a wrench applied at a different point (most importantly, with respect to a different coordinate frame).

To do this, let's consider a rigid body undergoing a transformation $g_{ab}(t)$. Here, A is our spatial frame, and B is our body frame (rigidly attached to the rigid body). The instantaneous body velocity is then represented by the twist ξ_{ab}. Notationally we could represent this as ξ_{ab}^b to denote the body velocity, but for ease of notation we will just write ξ_{ab}.

The amount of *work* done to the body by the wrench is defined by:

$$W = f \cdot d,$$

with d being the magnitude of the displacement caused by the force. Similarly, the amount of work done to the body by a torque is:

$$W = \tau \cdot \theta$$

We can combine these two equations using our wrench representation:

$$W = \begin{bmatrix} f & \tau \end{bmatrix} \cdot \begin{bmatrix} d \\ \theta \end{bmatrix}$$

However, we usually don't know the displacement or rotation caused from forces, but we can measure the velocity of the body. So to get to a velocity representation, we can use the following.

Consider the path integral definition of work:

$$W = \int_C f \cdot ds + \int_C \tau \cdot d\theta$$

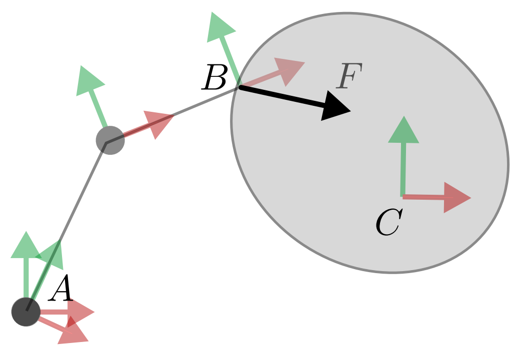
with C denoting the path caused by the force and ds being an infintesimal displacement along the path and $d\theta$ being the infintesimal rotation. We can translate this notion of a path into a time-derivative using our velocities:

$$\begin{aligned} W &= \int_{t_1}^{t_2} f \cdot v dt + \int_{t_1}^{t_2} \tau \cdot \omega dt \\ &= \int_{t_1}^{t_2} \begin{bmatrix} f \\ \tau \end{bmatrix} \cdot \begin{bmatrix} v \\ \omega \end{bmatrix} dt \\ &= \int_{t_1}^{t_2} \xi_b^\top F_b \, dt \\ \delta W &= \xi_b^\top F_b \end{aligned}$$

with F_b denoting the *wrench* applied to the body relative to the body frame.

We can now use this formula to equate whether or not wrenches are *equivalent*.

To contextualize this a bit better, consider the following example:



A wrench F_b is applied at the origin of coordinate frame B. We would like to find the equivalent wrench applied to the origin of the object's frame (frame C). We can compute this equivalent wrench as the wrench that results in the same instantaneous work:

$$\begin{aligned} \xi_c^\top F_c &= \xi_b^\top F_b \\ &= (\mathrm{Ad}_{g_{bc}} \xi_c)^\top F_b \\ &= \xi_c^\top \mathrm{Ad}_{g_{bc}}^\top F_b \\ F_c &= \mathrm{Ad}_{g_{bc}}^\top F_b \end{aligned}$$

In general, this leads to the change of coordinate frames for wrenches:

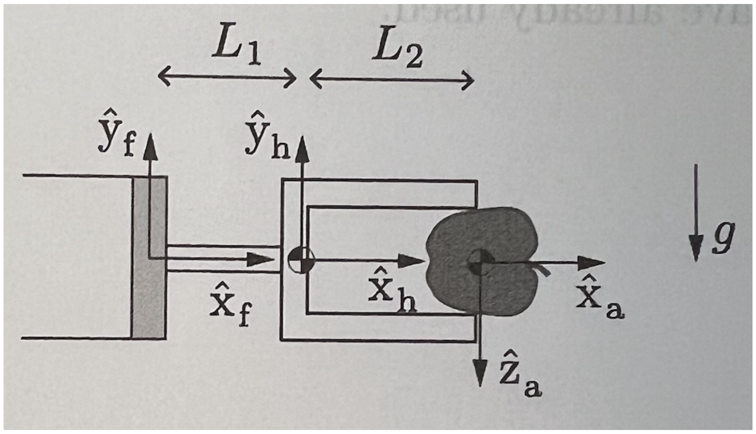
$$\begin{aligned} F_a &= \mathrm{Ad}_{g_{ba}}^\top F_b \\ F_b &= \mathrm{Ad}_{g_{ab}}^\top F_a \end{aligned}$$

Notice that this is similar to how we used adjoints to change the frame of reference for twists:

$$\xi_a = \mathrm{Ad}_{g_{ab}} \xi_b$$

Example

Consider the following end-effector (Example 3.28 in LP). Here, the end-effector is holding an apple with a mass of 0.1kg, in a gravitational field of $g = 10$m/s^2 (for simple math) acting downward on the page. The mass of the end-effector is 0.5kg. Lastly, assume $L_1 = 10$cm, and $L_2 = 15$cm.



Question: What is the force and torque measured by the six-axis force-torque sensor between the end-effector and the robot arm?

$$f_{\text{apple}} = \begin{bmatrix} 0 \\ 0 \\ 0.1 \cdot 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} N$$

$$f_{hand} = \begin{bmatrix} 0 \\ -0.5 \cdot 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix} N$$

To express these using wrenches:

$$F_a = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_h = \begin{bmatrix} 0 \\ -5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since we want the wrenches in the frame of the sensor (f), we can use the adjoint transformation to change the frame of reference for the wrenches:

$$F_f = \mathrm{Ad}_{g_{af}}^{\top} F_a$$

and

$$F_f = \mathrm{Ad}_{g_{hf}}^{\top} F_h$$

Once these wrenches are transformed into the same frame, we can combine them together to get the total wrench:

$$F_f = \mathrm{Ad}_{g_{af}}^{\top} F_a + \mathrm{Ad}_{g_{hf}}^{\top} F_h$$

To do this, let's first obtain our transformation matrices:

$$g_{af} = \begin{bmatrix} R_x(-\pi/2) & \begin{bmatrix} -L_1 - L_2 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -0.25m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{hf} = \begin{bmatrix} I & \begin{bmatrix} -L_1 \\ 0 \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -0.1m \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, plugging these into our adjoint transformations:

$$
\begin{aligned}
F_f &= \mathrm{Ad}^\top_{g_{af}} F_a + \mathrm{Ad}^\top_{g_{hf}} F_h \\
&= \left(g^\top_{af} \hat{F}_a (g^\top_{af})^{-1} \right)^\vee + \left(g^\top_{hf} \hat{F}_h (g^\top_{hf})^{-1} \right)^\vee \\
&\text{or} \\
&= \begin{bmatrix} R_{af} & [d_{af}]_\times R_{af} \\ 0 & R_{af} \end{bmatrix}^\top F_a + \begin{bmatrix} R_{hf} & [d_{hf}]_\times R_{hf} \\ 0 & R_{hf} \end{bmatrix}^\top F_h \\
&= \begin{bmatrix} 0 \\ -5N \\ 0 \\ 0 \\ 0 \\ -0.5Nm \end{bmatrix} + \begin{bmatrix} 0 \\ -1N \\ 0 \\ 0 \\ 0 \\ -0.25Nm \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -6N \\ 0 \\ 0 \\ 0 \\ -0.75Nm \end{bmatrix}
\end{aligned}
$$

Therefore, the sensor will read a force in the $y-$axis of -6N and a torque about the $z-$axis of -0.75Nm.

Example MATLAB Code

```
skew = @(m) [0, -m(3), m(2); m(3), 0, -m(1); -m(2), m(1), 0];
unskew = @(s) [s(3,2);s(1,3);s(2,1)];
hat = @(vector) [skew(vector(4:6)), vector(1:3); 0 0 0 0];
unhat = @(hattedv) [hattedv(1:3,4); unskew(hattedv(1:3,1:3))];
adjoint = @(g,F) g*hat(F)*inv(g);
admatrix = @(g) [g(1:3,1:3), skew(g(1:3,4))*g(1:3,1:3); ...
                 zeros(3,3), g(1:3,1:3)];

Fh = [0;-5;0; 0;0;0];
Fa = [0;0;1;0;0;0];

ghf = [eye(3), [-0.1;0;0]; 0 0 0 1];
gaf = [rotx(-90), [-0.25;0;0]; 0 0 0 1];


Ff1 = unhat(adjoint(ghf', Fh))
Ff2 = unhat(adjoint(gaf', Fa))

Ff1_alt = admatrix(ghf)'*Fh
Ff2_alt = admatrix(gaf)'*Fa
```

Relation to Jacobians

Lastly, torque and wrenches can be related using the Jacobian:

$$\tau = J_b^\top(\theta) F_b = J_s^\top(\theta) F_s$$

These equations relate the end-effector wrench to the joint torques by giving the torques that are equivalent to a wrench applied at the end-effector.

This relationship also stems from our definition of work:

$$\begin{aligned} W &= \int_{t_1}^{t_2} \tau \cdot \dot{\theta} dt = \int_{t_1}^{t_2} F \cdot \xi dt \\ \dot{\theta}^\top \tau &= \xi^\top F \\ &= (J_b \dot{\theta})^\top F_b && \text{(or } J_s \text{ and } F_s\text{)} \\ &= \dot{\theta}^\top J_b^\top F_b \\ \tau &= J_b^\top F_b \end{aligned}$$

These formulas can be used to address two separate questions:

1. If we apply an end-effector force, what torques are required to resist that force?
2. If we apply a set of joint torques, what is the resulting end-effector wrench?