Topics Covered:

- Introduction to Jacobians
- The Manipulator Jacobian
- Example

Additional Reading:

- LP Chapter 5

Introduction to Jacobians

The Jacobian is used in the context of robotics to relate end-effector velocity to joint velocity as a function of joint variables.

Definition: Jacobian

Assume we have a manipulator with coordinates $x \in \mathbb{R}^m$, velocity $\dot{x} = dx/dt \in \mathbb{R}^m$, and joint variables $\theta \in \mathbb{R}^n$. The forward kinematics can be written as:

$$x(t) = f(\theta(t)).$$

Using the chain rule, the time derivative at time t is:

$$\begin{aligned}\dot{x}(t) &= \frac{d}{dt} f(\theta(t)) \\ &= \frac{\partial f}{\partial \theta}\frac{d\theta}{dt} \\ &= J(\theta)\dot{\theta}\end{aligned}$$

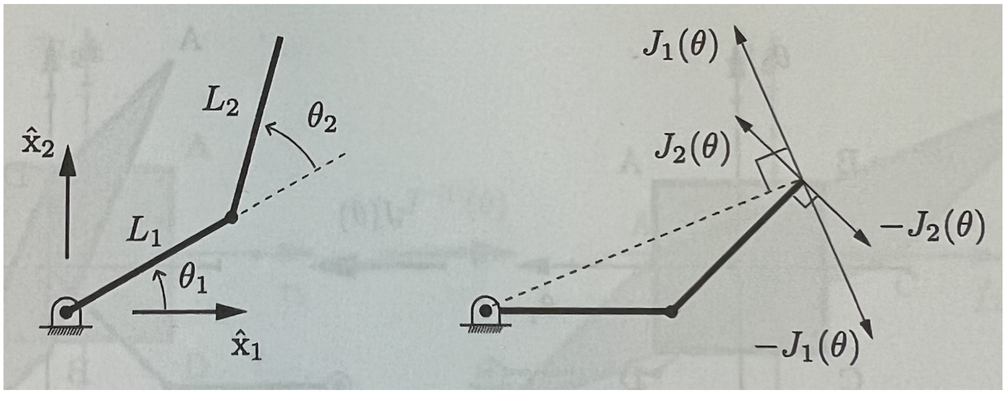
Here, $J(\theta) \in \mathbb{R}^{m \times n}$ is the Jacobian. This Jacobian matrix represents the linear sensitivity of the end-effector velocity $\dot{x}$ to the joint velocity $\dot{\theta}$ as a function of the joint variables θ. Explicitly, the Jacobian is:

$$J(\theta) := \frac{\partial f(\theta)}{\partial \theta} = Df(\theta) = \begin{bmatrix} \frac{\partial f_1(\theta)}{\partial \theta_1} & \cdots & \frac{\partial f_1(\theta)}{\partial \theta_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(\theta)}{\partial \theta_1} & \cdots & \frac{\partial f_m(\theta)}{\partial \theta_n} \end{bmatrix}$$

for $\theta = [\theta_1, \ldots, \theta_n]^\top$ and $f(\theta) = [f_1(\theta), \ldots, f_m(\theta)]^\top$.

Example

Consider the following example:



The explicit forward kinematics is defined as:

$$x_1 = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2)$$
$$x_2 = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)$$

Differentiating both sides yields:

$$\dot{x}_1 = -L_1 \sin(\theta_1)\dot{\theta}_1 - L_2 \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$
$$\dot{x}_2 = L_1 \cos(\theta_1)\dot{\theta}_1 + L_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$

Rearranging this to math the form $\dot{x} = J(\theta)\dot{\theta}$, we get:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} | & | \\ J_1(\theta) & J_2(\theta) \\ | & | \end{bmatrix}}_{J(\theta)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \end{aligned}$$

Therefore, the velocity of the end-effector canbe written in terms of the Jacobian and the joint velocities:

$$v_{tip} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

We can also see that we would have obtained the same thing by directly computing the Jacobian of the forward kinematics map: $Df(\theta)$:

$$\begin{aligned} Df(\theta) = J(\theta) &:= \begin{bmatrix} \frac{\partial f_1(\theta)}{\partial \theta_1} & \frac{\partial f_1(\theta)}{\partial \theta_2} \\ \frac{\partial f_2(\theta)}{\partial \theta_1} & \frac{\partial f_2(\theta)}{\partial \theta_2} \end{bmatrix} \\ &= \begin{bmatrix} -L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \\ &= \begin{bmatrix} J_1 & J_2 \end{bmatrix} \end{aligned}$$

One special aspect of the Jacobian is that it becomes a singular matrix when $J_1(\theta)$ and $J_2(\theta)$ become collinear (i.e., the two columns are linearly dependent, causing the Jacobian to lose rank). Thus, we can make conclusions about the *singularities* of the manipulator by looking at the Jacobian. These singularities are the configurations where the robot tip is unable to generate velocities in certain directions.

The Jacobian also relates to joint torque through the equation (we will discuss this further in Lecture 22):

$$f_{\text{tip}}^{\top} v_{\text{tip}} = \tau^{\top} \dot{\theta}$$

which can be transformed into the expression:

$$\begin{aligned} \tau &= J^{\top}(\theta) f_{\text{tip}} \\ f_{\text{tip}} &= J^{-1}(\theta) \tau \end{aligned}$$

This notion of a *Jacobian* is a generic definition available in the literature. When applied to a manipulator, since this Jacobian is often used to express the relationship between the end-effector coordinates and the joint configuration, it is sometimes called the *coordinate* Jacobian.

The Manipulator Jacobian

The *Manipulator Jacobian* specifically relates a six-dimensional twist ξ to the joint velocities. There are two standard Manipulator Jacobians, the *Spatial Manipulator Jacobian* and the *Body Manipulator Jacobian*. Each column of the spatial Jacobian corresponds to a screw axis expressed in the fixed frame with the screw axes depending on the joint variables θ. Each column of the body Jacobian corresponds to a screw axis expressed in the end-effector body frame. Note that before, for forward kinematics, our screw axes were always for the case $\theta = 0$.

We will derive how to express it with either the Product of Lie Groups or the Product of Exponentials.

Body and Spatial Manipulator Jacobians:

We will derive these expressions over the next few lectures. But in short, we have two different manipulator Jacobians of interest: the Spatial Jacobian which maps joint velocity $\dot{\theta}$ to the end-effectors spatial twist ξ^s, and the Body Jacobian which maps joint velocity $\dot{\theta}$ to the end-effectors body twist ξ^b. These are the same twists as defined in Lectures 6 and 7.

For the following expressions, recall that the *twist* (sometimes called Lie-algebra element, sometimes called the screw) has the form:

$$\xi_i = \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} = \begin{cases} \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix} & \text{if revolute} \\ \\ \begin{bmatrix} v_i \\ 0 \end{bmatrix} & \text{if prismatic} \end{cases}$$

The physical meaning here for ξ_i is that it describes the twist of the i^{th} joint in terms of the fixed frame with the robot in its zero position (i.e., $q_i = p_i^s(0)$). Sometimes this is also referred to as the screw-axis describing the i^{th} joint.

The *spatial manipulator Jacobian* $J^s(\theta) \in \mathbb{R}^{6\times n}$ associated with $\theta \in \mathbb{R}^n$ is defined as:

$$\begin{aligned} \xi^s &= J^s(\theta)\dot{\theta} \\ &= \begin{bmatrix} \xi_1 & \xi_2' & \cdots & \xi_n' \end{bmatrix} \dot{\theta} \\ &= \begin{bmatrix} \left(\frac{\partial g_e}{\partial \theta_1} g_e^{-1}\right)^\vee & \cdots & \left(\frac{\partial g_e}{\partial \theta_n} g_e^{-1}\right)^\vee \end{bmatrix} \dot{\theta} \\ &= \begin{bmatrix} \xi_1 & \mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}} \xi_2 & \cdots & \mathrm{Ad}_{e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{n-1}\theta_{n-1}}} \xi_n \end{bmatrix} \dot{\theta} \end{aligned}$$

Here, ξ_i' to denote the twist evaluated at the current configuration of the robot, *not* the zero configuration (i.e., $q_i = p_i^s(\theta)$).

The *body manipulator Jacobian*, $J^b(\theta) \in \mathbb{R}^{6\times n}$ is defined as:

$$\begin{aligned} \xi^b &= J^b(\theta)\dot{\theta} \\ &= \begin{bmatrix} \xi_1^\dagger & \xi_2^\dagger & \cdots & \xi_n^\dagger \end{bmatrix} \dot{\theta} && (\dagger \text{ is } \backslash\text{dagger}) \\ &= \begin{bmatrix} \left(g_e^{-1} \frac{\partial g_e}{\partial \theta_1}\right)^\vee & \cdots & \left(g_e^{-1} \frac{\partial g_e}{\partial \theta_n}\right)^\vee \end{bmatrix} \dot{\theta} \\ &= \begin{bmatrix} \mathrm{Ad}^{-1}_{e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_0} \xi_1 & \cdots & \mathrm{Ad}^{-1}_{e^{\hat{\xi}_n\theta_n} g_0} \xi_n \end{bmatrix} \dot{\theta} \end{aligned}$$

Here, $\xi_i^\dagger$ are the joint twists written with respect to the tool frame at the current configuration.

These Jacobians are related by the equation:

$$J^s(\theta) = \mathrm{Ad}_{g_e(\theta)} J^b(\theta)$$

Note: The spatial velocity will be particularly useful for inverse kinematics. For example, you can compute the joint velocities required to achieve a desired end-effector velocity if J^s is invertible using the expression:

$$\dot{\theta} = J^s(\theta)^{-1}\xi^s$$

where ξ^s is computed as the "unhatted" version of the homogenous twist $\hat{\xi}^s$, which can be computed using the expression $\hat{\xi}^s = \dot{g}g^{-1}$.

Similarly, if we want to compute the linear velocity of the end-effector, we can use the formula:

$$\dot{p}^s = \left(J^s(\theta)\dot{\theta}\right)^{\wedge} p^s$$
$$\dot{p}^b = \left(J^b(\theta)\dot{\theta}\right)^{\wedge} p^b$$

with p^s being a point attached to the frame of the end-effector, relative to the base frame, and p^b being a point attached to the frame of the end-effector, relative to the end-effector frame.

Lastly, we can still relate torques to the end-effector forces using the equation:

$$\tau = J^{s\top}(\theta) f^s_{\text{tip}} = J^{b\top}(\theta) f^b_{\text{tip}}$$

Example Revisted

So let's revisit the previous example. Let's assume that we want to solve for the spatial jacobian J^s:

One way we introduced of solving for the spatial manipulator Jacobian was through the expression:

$$J^s = \begin{bmatrix} \xi_1 & \xi_2' \end{bmatrix}$$

where ξ_i' indicates the twist evaluated at the current configuration of the robot, *not* the zero configuration as before. To do this, we will still use our formula for twists:

$$\xi_i = \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}$$

In our manipulator, both joint axes are aligned with the z-axis:

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The points on the axes (as a function of the joint variables at the current configuration and *not* the zero configuration) are:

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} L_1 \cos(\theta_1) \\ L_1 \sin(\theta_1) \\ 0 \end{bmatrix}$$

Then, the twists are:

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2' = \begin{bmatrix} L_1 \sin(\theta_1) \\ -L_1 \cos(\theta_1) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the spatial manipulator Jacobian is:

$$\begin{aligned} J^s(\theta) &= \begin{bmatrix} \xi_1 & \xi_2' \end{bmatrix} \\ &= \begin{bmatrix} 0 & L_1 \sin(\theta_1) \\ 0 & -L_1 \cos(\theta_1) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Note that we would get the same result by using either

$$J^s = \begin{bmatrix} \xi_1 & \mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1}} \xi_2 \end{bmatrix}$$

or

$$J^s = \begin{bmatrix} \left(\frac{\partial g_e}{\partial \theta_1} g_e^{-1} \right)^\vee & \left(\frac{\partial g_e}{\partial \theta_2} g_e^{-1} \right)^\vee \end{bmatrix}$$

Lastly, as a check, we can compute the linear velocity of the end-effector using the formula:

$$\begin{bmatrix} \dot{q}^s \\ 0 \end{bmatrix} = \xi^s \begin{bmatrix} q^s \\ 1 \end{bmatrix} = (J^s \dot{\theta})^\wedge \begin{bmatrix} q^s \\ 1 \end{bmatrix}$$

where $\dot{q}^s$ is the linear velocity of point q in the fixed spatial frame (q^s).

In our example, the point of our end-effector is $p = g_0 g_1 \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}$

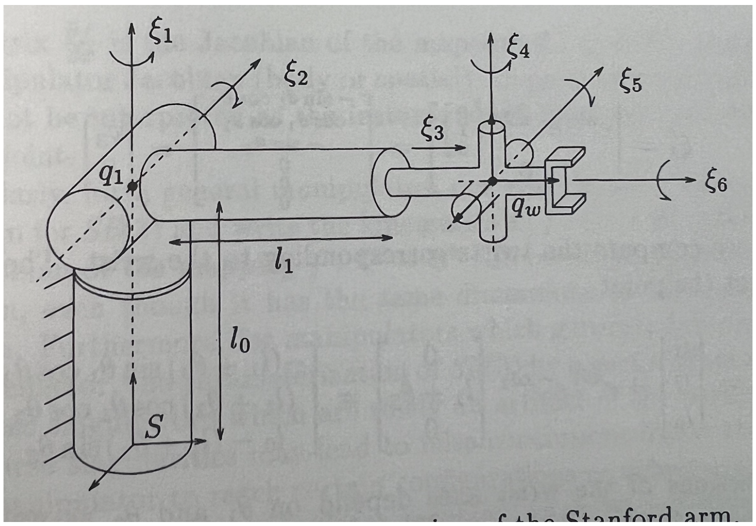
Finally, we can solve for the linear velocity at the end-effector as:

$$\begin{aligned} \begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} &= \left(\begin{bmatrix} 0 & L_1 \sin(\theta_1) \\ 0 & -L_1 \cos(\theta_1) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \right)^\wedge \begin{bmatrix} p \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} L_1 \sin(\theta_1)\dot{\theta}_2 + L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\ L_1 \cos(\theta_1) - L_1 \cos(\theta_1)\dot{\theta}_2 + L_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

While this expression is slightly different, it will give you the same result as before!

Example 2

Consider the following manipulator:



Lets first calculate the forward-kinematics using the product of exponentials (just for practice and comparison purposes):

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\ l_0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi_3 = \begin{bmatrix} v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(v_i is a unit vector pointing along the translational axis)

$$\xi_4 = \begin{bmatrix} -\omega_4 \times q_4 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} l_1 + \theta_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\xi_5 = \begin{bmatrix} -\omega_5 \times q_5 \\ \omega_5 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 \\ -l_0 \\ l_1 + \theta_3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi_6 = \begin{bmatrix} -\omega_6 \times q_6 \\ \omega_6 \end{bmatrix} = \begin{bmatrix} -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ l_0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} -l_0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Lastly, our g_0 is:

$$g_0 = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ l_0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}$$

Thus, our forward kinematics are:

$$g_e = e^{\xi_1\theta_1} \cdots e^{\xi_6\theta_6} g_0 \tag{1}$$

Now, let's compute the spatial manipulator Jacobian for this manipulator using the formula:

$$J^s = \begin{bmatrix} \xi_1 & \xi_2' & \xi_3' & \xi_4' & \xi_5' & \xi_6' \end{bmatrix}$$

For the first two joint axes, the point along the axes does not change with joint angles (i.e., $q_1 = [0, 0, l_0]^\top$). However, ω_2 will now be:

$$\omega_2' = R_z(\theta_1)\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos(\theta_1) \\ -\sin(\theta_1) \\ 0 \end{bmatrix}$$

Also, the prismatic joint will be:

$$v_3' = R_z(\theta_1)R_x(-\theta_2)\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -s_1c_2 \\ c_1c_2 \\ -s_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Next, the wrist location (q_w') will be:

$$q_w' = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} + R_z(\theta_1)R_x(-\theta_2)\begin{bmatrix} 0 \\ l_1 + \theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -(l_1+\theta_3)s_1c_2 \\ (l_1+\theta_3)c_1c_2 \\ l_0 - (l_1+\theta_3)s_2 \end{bmatrix}$$

Lastly, the final three joint axes will be:

$$\begin{aligned}
\omega_4' &= R_z(\theta_1)R_x(-\theta_2)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_1s_2 \\ c_1s_2 \\ c_2 \end{bmatrix} \\
\omega_5' &= R_z(\theta_1)R_x(-\theta_2)R_z(\theta_4)\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1c_4 + s_1c_2s_4 \\ -s_1c_4 - c_1c_2s_4 \\ s_2s_4 \end{bmatrix} \\
\omega_6' &= R_z(\theta_1)R_x(-\theta_2)R_z(\theta_4)R_x(-\theta_5)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_5(s_1c_2c_4 + c_1s_4) + s_1s_2s_5 \\ -c_5(c_1c_2c_4 - s_1s_4) - c_1s_2s_5 \\ -s_2c_4c_5 - c_2s_5 \end{bmatrix}
\end{aligned}$$

All of these updated parameters would then be plugged into the following formula for the complete spatial manipulator Jacobian:

$$J^s = \begin{bmatrix} 0 & -\omega_2' \times q_1 & v_3' & -\omega_4' \times q_w' & -\omega_5' \times q_w' & -\omega_6' \times q_w' \\ \omega_1 & \omega_2' & 0 & \omega_4' & \omega_5' & \omega_6' \end{bmatrix}$$