

Topics Covered:

- Introduction to Inverse Kinematics
- Geometric Approach to Inverse Kinematics

Additional Reading:

- Craig Chapter 4; LP Chapter 6; MLS Chapter 3, Section 3

Introduction to Inverse Kinematics

Inverse kinematics refers to the process of determining a feasible joint-configuration given a desired end-effector configuration.

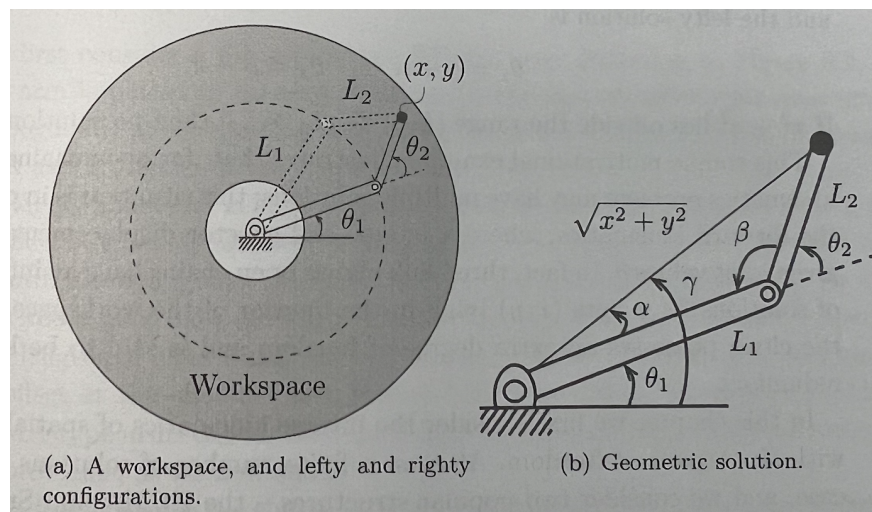
- this is a nonlinear problem
- clearly desired end-effector configuration must be in workspace to have a solution (sometimes will have multiple solutions)
- no real general algorithms exist to handle the nonlinear inverse kinematics problem
- problem/manipulator considered solveable if joint-configuration can be determined by an algorithm that allows one to determine all joint variables associated with a given end-effector configuration.
- Main approaches:
 - Analytic Inverse Kinematics (find solutions satisfying $g_e(\vec{\theta}) = g_e^d$)
 - * Geometric approach (intuitive, not systematic, not always applicable)
 - * There are several optimization-based algorithms that solve for the solutions such as IKFast
 - Numerical Inverse Kinematics (iterative methods to minimize error $||g_e(\vec{\theta}) - g_e^d||$)
 - * Iteratively find the roots of a nonlinear equation using nonlinear root-finding methods (Newton-Raphson)
 - Paden-Kahan (systematic algorithm that works for certain manipulators)
 - Pieper's solution (closed form solution for a serial 6 DOF in which three consecutive axes intersect at a point (including robots with three consecutive parallel axes since they meet at a point at infinity). Pieper's method applies to the majority of commercially available industrial robots.)

The Geometric Approach

- by observation, reduce geometry of the linkage to easily solved sub-problems.
- for example, one technique reduces the linkage geometry to triangles

Example

Many manipulators have the following subproblem, which is usually obtained by projecting the manipulator geometry to 2 coordinate axes.



Here, the forward kinematics are given by:

$$\begin{Bmatrix} x_c \\ y_c \\ \theta_c \end{Bmatrix} = \begin{Bmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \\ \theta_1 + \theta_2 \end{Bmatrix}$$

Assuming $L_1 > L_2$, the set of reachable points (the workspace) is an annulus of inner radius $L_1 - L_2$ and outer radius $L_1 + L_2$. Looking at the workspace above, we can visually see that there will be either zero, one, or two solutions depending on whether (x, y) lies on the exterior, boundary, or interior of this annulus, respectively.

When there are two solutions, the elbow joint (second joint) will be either positive or negative. This is sometimes called “lefty” or “righty”, or “elbow-up” or “elbow-down” solutions.

Here, orientation is not fully controllable, but consider placement of end-effector in Euclidean space. The manipulator (port) forms a triangle:

CHECK: $l_1 + l_2 < \sqrt{x_e^2 + y_e^2}$

As long as this condition is satisfied, we can use the law of cosines for our inverse kinematics. Recall that the law of cosines is:

$$c^2 = a^2 + b^2 - 2ab \cos(C)$$

where a , b , and c are the sides of a triangle and C is the angle between sides a and b (opposite of side c).

In our situation, the law of cosines yields the formula:

$$x^2 + y^2 = l_1^2 + l_2^2 - 2l_1l_2 \cos(\beta)$$

Thus, rearranging this equation gives us:

$$\beta = \cos^{-1} \left(\frac{l_1^2 + l_2^2 - x^2 - y^2}{2l_1l_2} \right)$$

Similarly, applying the law of cosines to α gives us:

$$\alpha = \cos^{-1} \left(\frac{x^2 + y^2 + l_1^2 - l_2^2}{2l_1\sqrt{x^2 + y^2}} \right)$$

However, there is a little problem since inverse of cosine is not unique (there are 2 ways to get x_{des} and y_{des}). So there will be only 2 out of 4 possible solutions which actually work. We will find these solutions by verifying the orientation at the end.

Taking γ using the two-argument arctangent function:

$$\gamma = \text{atan2}(y, x),$$

we can solve for the joint angles (assuming the righty-solution) as:

$$\begin{aligned} \theta_1 &= \gamma - \alpha \\ \theta_2 &= \pi - \beta \end{aligned}$$

The lefty solution is:

$$\begin{aligned} \theta_1 &= \gamma + \alpha \\ \theta_2 &= \beta - \pi \end{aligned}$$

Lastly, we solve for orientation as $\theta_1 + \theta_2$. This orientation must be equal to:

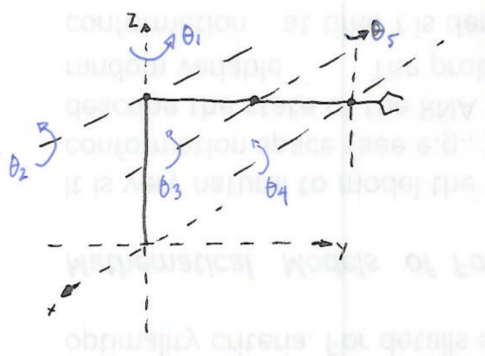
$$\begin{aligned} \theta_1 + \theta_2 &= \gamma - \alpha + \pi - \beta \\ &= \gamma + \alpha + \beta - \pi \end{aligned}$$

We will have to check these conditions to make sure that the solution we found is valid.

But, how can geometric technique be applied to more complicated manipulators? We will break the problem down to:

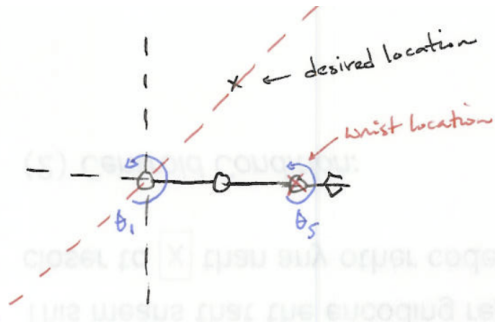
1. position of the wrist
2. orientation of the hand/wrist
3. angle of the first joint

We will assume that (2) is solved enough to give us desired position of the wrist.



1. has wrist at end for reorientation
2. to place wrist at a particular location, consider the following projections

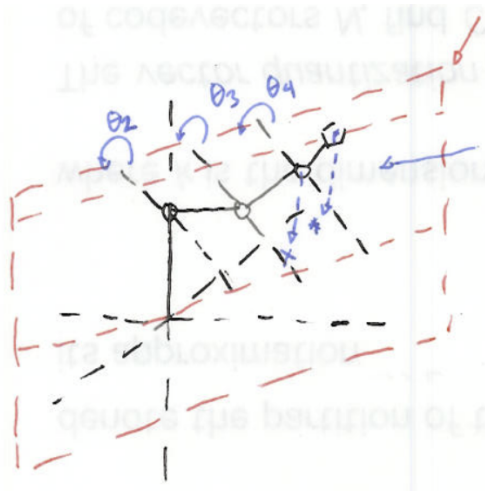
1. Top View



If we can get the manipulator to lie along the red dotted line, then we can project to this plane since all of the remaining joints work in this plane.

Basically, $\theta_1 = \text{atan2}(x, y) - \frac{\pi}{2}$ (since reference configuration aligned with y-axis and not x-axis)

2. Side View



red arrow: plane of projection
blue arrow: end-effector is too high and too far out (same with wrist)

When projected to the plane, the placement of the wrist reduces to the previous subproblem (θ_1, θ_2) .

Summary of the Geometric Approach

1. it's easy because you are working it out intuitively using geometry.
2. not systematic because requires good intuition
3. not always applicable (limited set of “geometric techniques”). Good thing is that design of manipulators can allow for simple/geometric solution.