

Topics Covered:

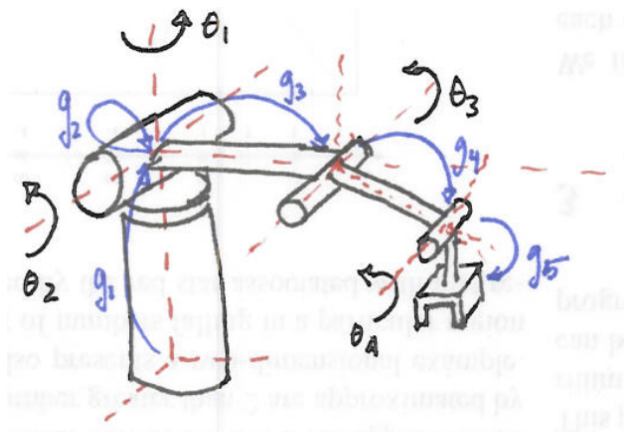
- Product of Exponentials
- Denavit Hartenberg Parameters

Additional Reading:

- LP 4.1; MLS Chapter 3, Section 2.2;

Review

Consider the following manipulator:



We've already learned how to use the product of homogeneous transformation matrices (also known as the Product of Lie Groups) to derive the forward kinematics $g_E = g_{01}g_{12}g_{23}g_{34}g_{4E}$. Specifically, we obtained the displacements:

$$d_{01} = \begin{Bmatrix} 0 \\ 0 \\ l_0 \end{Bmatrix}, \quad d_{12} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad d_{23} = \begin{Bmatrix} 0 \\ l_1 \\ 0 \end{Bmatrix}, \quad d_{34} = \begin{Bmatrix} 0 \\ l_2 \\ 0 \end{Bmatrix}, \quad d_{4E} = \begin{Bmatrix} 0 \\ 0 \\ -l_3 \end{Bmatrix}$$

and the rotations:

$$\begin{aligned}
 R_{01} &\rightarrow \text{rotation about z-axis} \\
 R_{01} &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 R_{23} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_3) & -\sin(\theta_3) \\ 0 & \sin(\theta_3) & \cos(\theta_3) \end{bmatrix} \\
 R_{4E} &= I \\
 R_{12} &\rightarrow \text{rotation about x-axis} \\
 R_{12} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_2) & -\sin(\theta_2) \\ 0 & \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \\
 R_{34} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_4) & -\sin(\theta_4) \\ 0 & \sin(\theta_4) & \cos(\theta_4) \end{bmatrix}
 \end{aligned}$$

This class, we will discuss two alternative methods: the product of exponentials and Denavit-Hartenberg parameters. These methods are more commonly used in practice.

Product of Exponentials

The product of exponentials uses the formula:

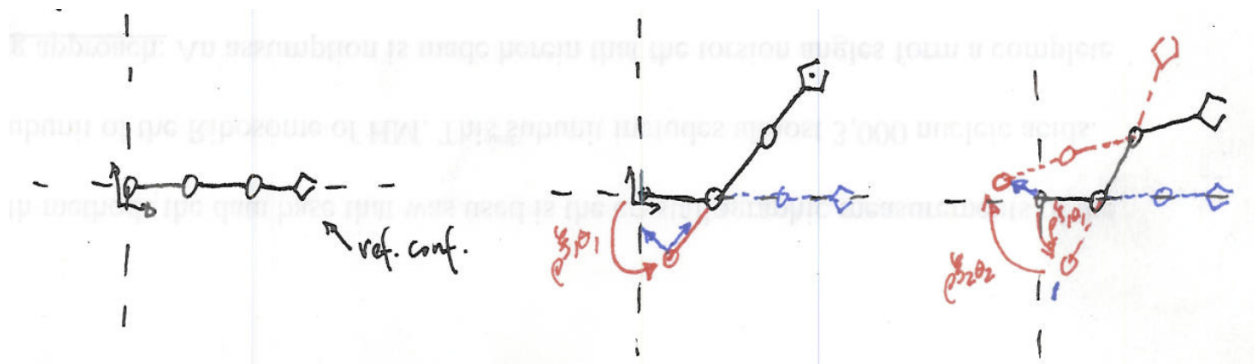
$$g_e = e^{\xi_1 \theta_1} \dots e^{\xi_n \theta_n} g_0.$$

Note here that if $\vec{\theta} = 0$, then $e^0 = I$, and thus $g_e(0) = g_0$. Therefore, g_0 encodes the displacement that occurs at the "zero configuration", and is known as the reference configuration for the manipulator.

A given ξ_i will then say how the reference configuration changes for a given θ_i ,

$$g_e(0, \dots, \theta_i, \dots, 0) = e^{\xi_i \theta_i} g_0$$

This can be visualized as:



What are the ξ_i ? There are three basic types corresponding to the 3 single degree of freedom lower-pair joints:

1) revolute:	$\xi_i = \begin{Bmatrix} -\omega_i \times q_i \\ \omega_i \end{Bmatrix}$	ω_i - unit vector aligned w/ rotation axis q_i - point on the axis
2) prismatic:	$\xi_i = \begin{Bmatrix} v_i \\ 0 \end{Bmatrix}$	v_i - unit vector aligned w/ translation axis
3) helical/screw:	$\xi_i = \begin{Bmatrix} h\omega_i + [q_i]_{\times}\omega_i \\ \omega_i \end{Bmatrix}$	h - pitch of helical motion

These $\xi_i \in \mathfrak{se}(3)$ represent twists in spatial coordinates. Note here that $-\omega_i \times q_i = q_i \times \omega_i = [q_i]_{\times} \omega_i$.

Based on seeing how end-effector changes for each joint from base to end-effector frame, can combine for all i to get total effect:

$$g_e = \underbrace{e^{\xi_1 \theta_1} \dots e^{\xi_n \theta_n}}_{\text{order matters!}} g_0$$

Order here matters! The (ξ_i, θ_i) must enumerate from base to tool frame.

Note: Matrix Exponentials do *not* commute (unless the actual matrices commute, i.e. $AB = BA$), this is why order matters for the Product of Exponentials:

$$e^A e^B \neq e^B e^A$$

Note about Computing Matrix Exponential by Hand

Remember, we can compute the matrix exponentials by hand using our previous formulas.

$$\exp(\hat{\xi}\theta) = \begin{bmatrix} e^{[\omega]_{\times}\theta} & (I - e^{\hat{\omega}\theta})[\omega]_{\times}\hat{v} + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

where $\exp([\omega]_{\times}\theta)$ is computed using Rodrigues' formula (and assuming $\|\omega\| = 1$):

$$e^{[\omega]_{\times}\theta} = I + \sin(\theta)[\omega]_{\times} + (1 - \cos(\theta))[\omega]_{\times}^2$$

However, as we saw with our computation of twists, this formula can be simplified if we know that our associated joint is a revolute joint. In this case, our exponential map simplifies to:

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{[\omega]_{\times}\theta} & (I - e^{[\omega]_{\times}\theta})q \\ 0 & 1 \end{bmatrix}$$

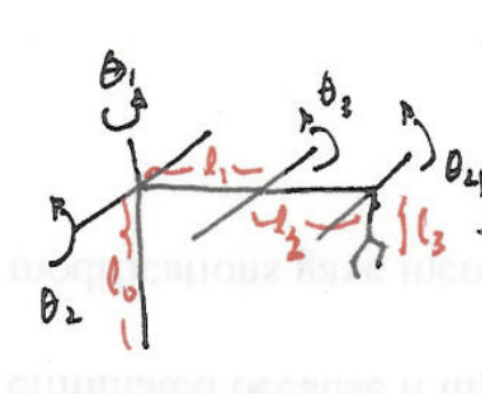
where q is the position vector from the origin to any point on the rotation axis associated with ω . And again, $\exp([\omega]_{\times}\theta)$ is computed using Rodrigues' formula. Alternatively, if the rotation axis is about a principle axis, this expression can be obtained using our traditional rotation matrices.

If the joint is prismatic, the exponential map simplifies to:

$$\exp(\hat{\xi}\theta) = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$$

Example

Let's consider the example from before. For clarity, let's redraw the zero configuration:



With this zero configuration, the reference configuration is:

$$g_e(0) = g_0 = \begin{bmatrix} I & \begin{Bmatrix} 0 \\ l_1 + l_2 \\ l_0 - l_3 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

Next, we need to find each twist ξ_i that we will plug into the expression:

$$g_e(\theta) = e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_4 \theta_4} g_0$$

What are the ξ_i ?

$\xi_1:$	(rotation axis) $\omega_1 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$, (point on axis) $q_1 = \begin{Bmatrix} 0 \\ 0 \\ l_0 \end{Bmatrix}$	$\xi_1 = \begin{Bmatrix} q_1 \times \omega_1 \\ \omega_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$
$\xi_2:$	$\omega_2 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$, $q_2 = \begin{Bmatrix} 0 \\ 0 \\ l_0 \end{Bmatrix}$, $q_2 \times \omega_2 = \begin{Bmatrix} 0 \\ l_0 \\ 0 \end{Bmatrix}$	$\xi_2 = \begin{Bmatrix} 0 \\ l_0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$
$\xi_3:$	$\omega_3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$, $q_3 = \begin{Bmatrix} 0 \\ l_1 \\ l_0 \end{Bmatrix}$, $q_3 \times \omega_3 = \begin{Bmatrix} 0 \\ l_0 \\ -l_1 \end{Bmatrix}$	$\xi_3 = \begin{Bmatrix} 0 \\ l_0 \\ -l_1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$
$\xi_4:$	$\omega_4 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$, $q_4 = \begin{Bmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{Bmatrix}$, $q_4 \times \omega_4 = \begin{Bmatrix} 0 \\ l_0 \\ -l_1 - l_2 \end{Bmatrix}$	$\xi_4 = \begin{Bmatrix} 0 \\ l_0 \\ -l_1 - l_2 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$

Then, to actually compute each matrix exponential, we could use our expressions for $e^{\hat{\xi}_i \theta_i}$:

$$e^{\hat{\xi}_i \theta_i} = \begin{bmatrix} \exp([\omega]_{\times} \tau) & (I - \exp([\omega]_{\times} \tau)) \frac{[\omega]_{\times} v}{\|\omega\|^2} + \frac{\omega \omega^T}{\|\omega\|^2} v \tau \\ 0 & 1 \end{bmatrix}$$

and $\exp([\omega]_{\times} \tau)$ obtained using Rodrigues' formula:

$$\exp([\omega]_{\times} \tau) = I + \frac{[\omega]_{\times}}{\|\omega\|} \sin(\|\omega\| \tau) + \frac{[\omega]_{\times}^2}{\|\omega\|^2} (1 - \cos(\|\omega\| \tau))$$

OR, if you have access to a computer, you can simply use the matrix exponential function in MATLAB (`expm()`) or Python (`scipy.linalg.expm()`) to compute these. This method is much simpler in practice and only requires you to compute the homogeneous form of ξ (i.e., $\hat{\xi}$) as:

$$\hat{\xi} = \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix}$$

with v calculated using the previous formulas. For our example, each of the homogeneous twists

are:

$$\begin{aligned}\hat{\xi}_1 &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \hat{\xi}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & l_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \hat{\xi}_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & l_0 \\ 0 & 1 & 0 & -l_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & \hat{\xi}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & l_0 \\ 0 & 1 & 0 & -l_1 - l_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

The benefit of the Product of Exponentials is that it is more general than the Product of Lie Groups and can more easily handle any arbitrary axis of rotation for the joint axes.

We can verify that it produces the same results by comparing to the product of Lie Groups:

$$\begin{aligned}g_e(\theta) &= e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_4\theta_4} g_0 \\ e^{\hat{\xi}_1\theta_1} &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ e^{\hat{\xi}_2\theta_2} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_2) & -\sin(\theta_2) & l_0 \sin(\theta_2) \\ 0 & \sin(\theta_2) & \cos(\theta_2) & l_0(1 - \cos(\theta_2)) \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ e^{\hat{\xi}_3\theta_3} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_3) & -\sin(\theta_3) & l_0 \sin(\theta_3) + l_1(1 - \cos(\theta_3)) \\ 0 & \sin(\theta_3) & \cos(\theta_3) & l_0(1 - \cos(\theta_3)) - l_1 \sin(\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ e^{\hat{\xi}_4\theta_4} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_4) & -\sin(\theta_4) & l_0 \sin(\theta_4) + (l_1 + l_2)(1 - \cos(\theta_4)) \\ 0 & \sin(\theta_4) & \cos(\theta_4) & l_0(1 - \cos(\theta_4)) - (l_1 + l_2) \sin(\theta_4) \\ 0 & 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

Multiplying this all out should give us the same solution as before. Note: your homework will be to implement this example.

Denavit-Hartenberg Parameters

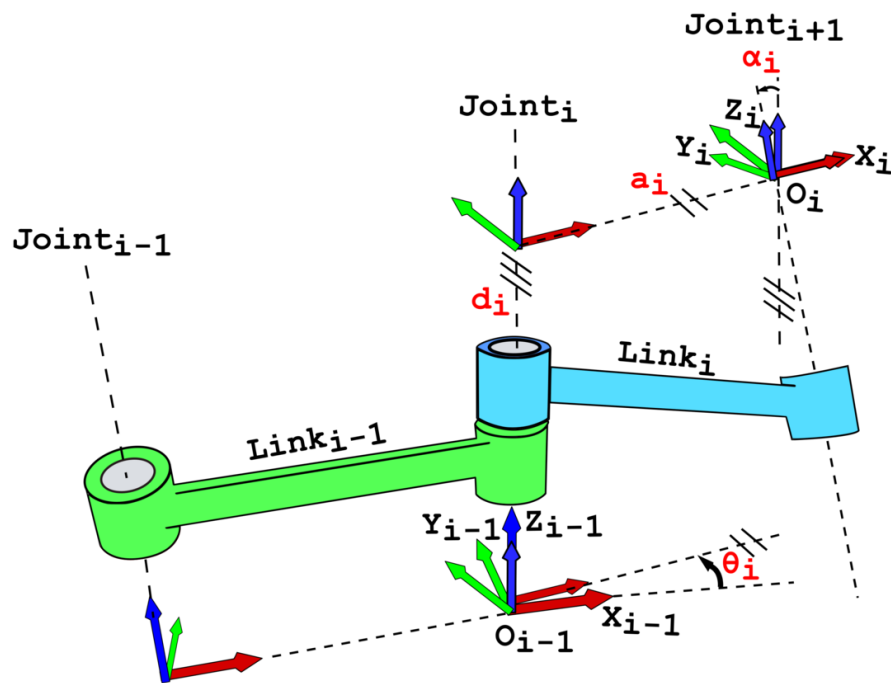
Lastly, the Denavit-Hartenberg parameters are a systematic way to describe the geometry of a manipulator. This method begins by assigning a specific frame of reference to each joint:

- the x-axis is parallel to the common normal (the line perpendicular to two non-intersecting joint axes)
- the z-axis is in the direction of the joint axis
- the y-axis follows from the x- and z-axes by choosing it to be a right-handed coordinate system

Using these reference frames, the method then assigns the following four parameters to each joint:

- a_i - distance from z_{i-1} to z_i along x_i axis
- α_i - angle from z_{i-1} to z_i about x_i axis
- d_i - distance from x_{i-1} to x_i along z_{i-1} axis
- θ_i - angle from x_{i-1} to x_i about z_{i-1} axis

For example, the diagram given by wikipedia is:



For our example, the parameters are as follows:

Link	a_i	α_i	d_i	θ_i
1	0	0	l_0	θ_1
2	l_1	$\pi/2$	0	θ_2
3	l_2	0	0	θ_3
4	l_3	0	0	$\theta_4 - \pi/2$

Once these parameters are assigned, the transformation matrix from frame $i - 1$ to frame i can be written as:

$$A_i = R_z(\theta_i)T_z(d_i)T_x(a_i)R_x(\alpha_i)$$

$$= \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\cos(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\cos(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

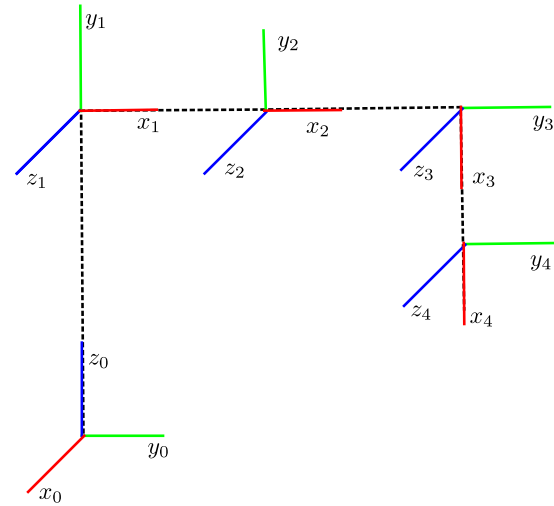
The forward kinematics can then be found by multiplying these matrices together:

$$g_E = A_1A_2A_3A_4$$

Thus, the Denavit-Hartenberg parameters for the previously presented manipulator (with the frames configured as shown below) are:

Table 1: Denavit-Hartenberg parameters for the example manipulator

Link	a_i	α_i	d_i	θ_i
1	0	$\pi/2$	l_0	$\theta_1 + \pi/2$
2	l_1	0	0	θ_2
3	l_2	0	0	$\theta_3 - \pi/2$
4	l_3	0	0	θ_4



where in every frame you:

1. Rotate about z_{i-1} by θ_i
2. Translate along z_{i-1} by d_i
3. Translate along x_i by a_i
4. Rotate about x_i by α_i

Note that there is also a modified version of the Denavit-Hartenberg parameters that switches the order of the translations and rotations. This is known as the Craig convention, and uses the

following transformation matrix:

$$A_i = R_x(\alpha_{i-1})T_x(a_{i-1})R_z(\theta_i)T_z(d_i)$$

$$= \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & a_{i-1} \\ \sin(\theta_i)\cos(\alpha_{i-1}) & \cos(\theta_i)\cos(\alpha_{i-1}) & -\sin(\alpha_{i-1}) & -d_i\sin(\alpha_{i-1}) \\ \sin(\theta_i)\sin(\alpha_{i-1}) & \cos(\theta_i)\sin(\alpha_{i-1}) & \cos(\alpha_{i-1}) & d_i\cos(\alpha_{i-1}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

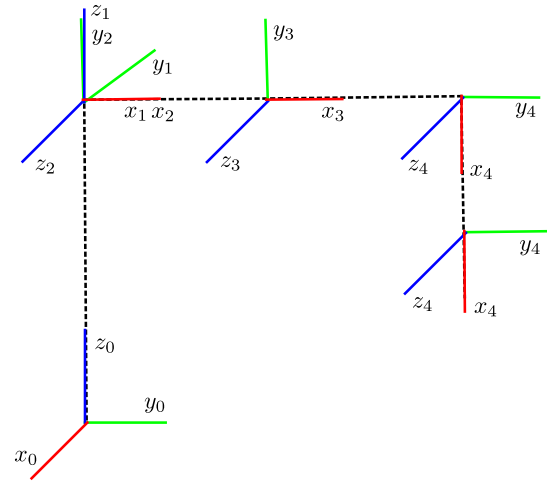
Here in every frame you now:

1. Rotate about x_{i-1} by α_{i-1}
2. Translate along x_{i-1} by a_{i-1}
3. Rotate about z_i by θ_i
4. Translate along z_i by d_i

The DH parameters for our example using the Craig convention are shown in the table below (with the associated frames):

Table 2: Denavit–Hartenberg parameters for the example manipulator

Link	a_{i-1}	α_{i-1}	d_i	θ_i
0	0	0	l_0	$\pi/2$
1	0	$\pi/2$	0	θ_1
2	l_1	0	0	θ_2
3	l_2	0	0	θ_3
4	l_3	0	0	$\theta_4 - \pi/2$



You can verify that this produces the same result as before. There are several convenient visualizers for DH parameters online, such as this one.