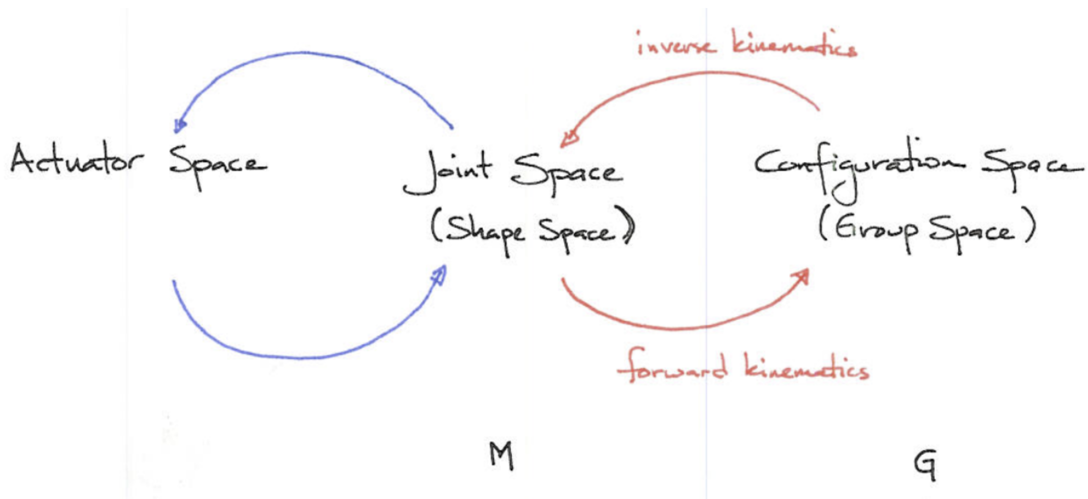
Topics Covered:

- Joints
- Workspace Description
- Forward Kinematics

Additional Reading:

- Craig 3.6 & 3.7, LP 2.2 & 2.3, MLS Chapter 3, Section 2

Manipulators & Manipulator Analysis In "Introduction to Robotics", Craig provides a nice visualization that illustrates the mappings between kinematic descriptions:

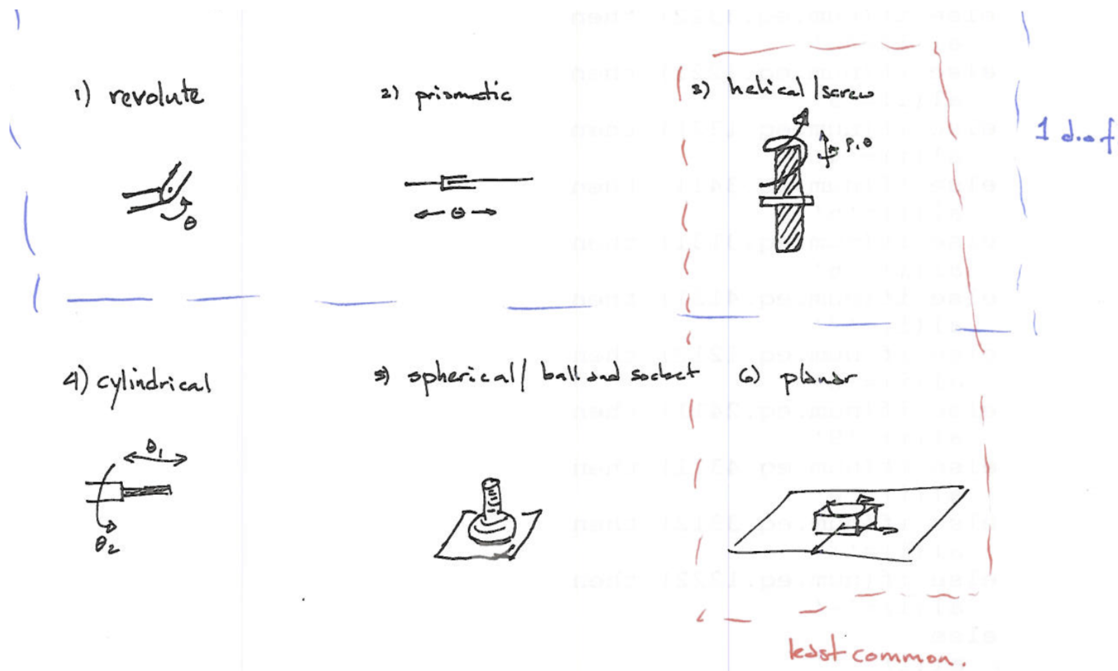


This diagram illustrates the three representations of a manipulator's position and orientation: descriptions in actuator space, joint space, and Cartesian space. Importantly, forward mappings can be constructed to map from actuator space to joint space, as well as from joint space to configuration space. Alternatively, inverse mappings are constructed to then map the reverse.

So far in this course, we have been working mostly with configuration space, with some forward kinematics. We will now delve deeper into different methods of forward kinematics, as well as then discussing inverse kinematics. Lastly, we will discuss the relationship between actuator and joint space (which is loosely the field of control).

Joint Space

Joints are traditionally chosen from a set of 6 simpler ones, called lower-pair joints,



Their simpler nature allows for product of exponentials formula (a method of forward kinematics which we will discuss next class).

If we consider each joint to have 1 DOF, then the joint space is products of

1. S^1 (revolute joint has the topology of a 1-dimensional sphere)
2. $\mathbb{R}$ (prismatic or helical joints have the topology of a 1-dimensional line)

Here, products of S^1 gives the torus T^r and products of $\mathbb{R}$ gives a p-dimensional Euclidean space:

$$T^r = \underbrace{S^1 \times \cdots \times S^1}_{r \text{ copies}}$$
$$\mathbb{R}^p = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{p \text{ copies}}$$

Together we can mathematically represent the joint space (also called the manipulator space) as:

$$M = T^r \times \mathbb{R}^p$$
r : # revolute joints
p : # prismatic/helical joints

Workspace Description

Definition: Workspace(Complete)

$$W = \{g_e(\overrightarrow{\theta}) \in SE(n) \mid \overrightarrow{\theta} \in M\}$$

The workspace W denotes the set of all configurations reachable by some joint configuration. It's usually difficult to interpret or visualize. Thus, an alternative is the reachable workspace.

Definition: Reachable Workspace

$$W_R = \{p_e(\overrightarrow{\theta}) \in E(n) \mid \overrightarrow{\theta} \in M\}$$

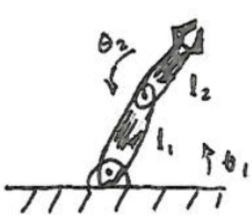
The reachable workspace W_R is the set of positions reachable by some joint configuration. It is a volume of $E(n)$ which can be reached at <u>some</u> orientation. Notably, this is not necessarily a useful measure since orientation is not always controllable.

Definition: Dextrous Workspace

$$W_D = \{p_e(\overrightarrow{\theta}) \in E(n) \mid \forall R \in SO(n), \exists \overrightarrow{\theta} \in M \text{ s.t. } g_e(\overrightarrow{\theta}) = (p_e, R)\}$$

The dextrous workspace W_D is the set of positions reachable with arbitrary orientation. Within this volume we can do anything. In other words, the end-effector has full rotational freedom at every point in this workspace.
Typically, to maximize dextrous workspace, industrial manipulators add a spherical wrist to the end of the manipulator chain. For example, SCARA manipulator adds a cylindrical joint for full $SE(2) \subset SE(3)$ control.

<u>Example 1. (Kinematically insufficient manipulator)</u>

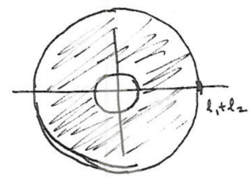


$$g_e(\theta) = \begin{Bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) \\ \theta_1+\theta_2 \end{Bmatrix}$$

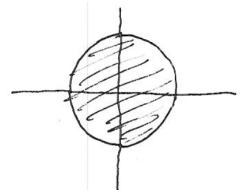
Note: dim(M) < dim(G)

W: will be a surface in SE(3); does not take up any volume (system is not fully controllable). Hard to visualize

W_R: (if $l_1 \neq l_2$) annulus $|l_1 - l_2| < r < l_1 + l_2$

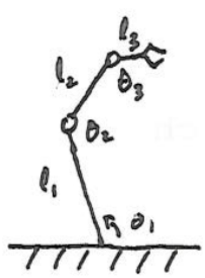


(if $l_1 = l_2 = l$) disc of radius $2l$



W_D: (if $l_1 \neq l_2$), then you get an empty set $\emptyset$
(if $l_1 = l_2 = l_1$), get origin only .

Example 2.

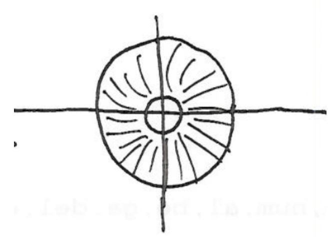


$$g_e(\theta) = \begin{Bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) \\ \theta_1+\theta_2+\theta_3 \end{Bmatrix}$$

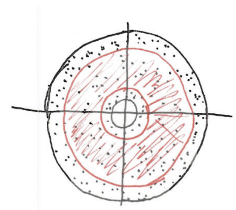
dim(M) = dim(G) $\Longrightarrow$ should have non-trivial dextrous workspace

assume: $l_1 > l_2 > l_3$
and $l_1 > l_2 > l_3$

W_R: annulus $l_1 - l_2 - l_3 \leq r \leq l_1 + l_2 + l_3$



W_D: annulus $l_1 - l_2 + l_3 \leq r \leq l_1 + l_2 - l_3$



loses $2l_3$ of inner and outer radii

Forward Kinematics

Forward kinematics is the process of mapping from joint space to configuration space. Typically, this is the process of calculating the configuration of end-effector given joint configuration of the manipulator.

Definition: Forward Kinematics

The forward kinematics of a manipulator is the configuration of the end-effector given a joint configuration of the manipulator.

We will discuss this in more detail in the next lecture, but there are many methods of doing this. All methods are done by concatenating transformations that go from joint-to-joint.

- Homogeneous Transformation Matrices: represents each joint's transformation (rotation and/or translation) using homogeneous transformation matrices

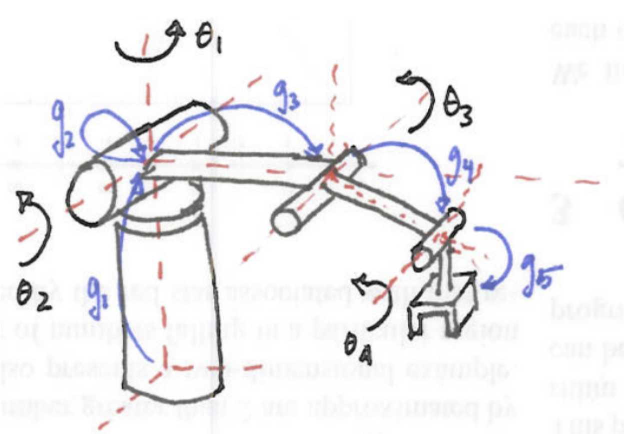
$$g_{we} = g_{wa}g_{ab}\cdots g_{ne}$$

- Product of Exponentials: uses the exponential of a twist (screw motion) to describe the movement of each joint

$$g(\theta) = e^{\hat{\xi}\theta_1}\cdots e^{\hat{\xi_n}\theta_n}g_0$$

- Denavit-Hartenberg parameters: simplifies the description of the robot's geometry by introducing four parameters for each joint
- Geometric Methods: uses basic geometry and trigonometry

Example:



← what are the forward kinematics? This is where benefits of Lie group notation come in.

Using what we have learned so far (Homogeneous Transformation Matrices), we can write the end-effector configuration as product of Lie-group elements: $g = g_{01}g_{12}g_{23}g_{34}g_{4E}$. Here, each g_{ij} goes from one joint to the next across links.

- getting displacement is easy
- getting orientation (rotation) is tricky

Recall, $SE(3) = E(3) \times SO(3)$ e.g. $g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, with $p \in E(3)$ and $R \in SO(3)$.

What are the R_{01}, R_{12}, etc.?

$$\begin{array}{cc}
R_{01} \to \text{rotation about z-axis} & R_{12} \to \text{rotation about x-axis} \\
R_{01} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} & R_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_2) & -\sin(\theta_2) \\ 0 & \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \\
R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_3) & -\sin(\theta_3) \\ 0 & \sin(\theta_3) & \cos(\theta_3) \end{bmatrix} & R_{34} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_4) & -\sin(\theta_4) \\ 0 & \sin(\theta_4) & \cos(\theta_4) \end{bmatrix} \\
R_{4E} = I &
\end{array}$$

In the next lecture, we will discuss further the Lie group $SO(3)$ and the representations for $SO(3)$ and therefore $SE(3)$.

Meanwhile, what about p_{01}, p_{12}, etc.?

$$p_{01} = \begin{Bmatrix} 0 \\ 0 \\ l_0 \end{Bmatrix}, \quad p_{12} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad p_{23} = \begin{Bmatrix} 0 \\ l_1 \\ 0 \end{Bmatrix}, \quad p_{34} = \begin{Bmatrix} 0 \\ l_2 \\ 0 \end{Bmatrix}, \quad p_{4E} = \begin{Bmatrix} 0 \\ 0 \\ -l_3 \end{Bmatrix}$$

Therefore,

$$\begin{aligned}
g_E &= g_{01} g_{12} g_{23} g_{34} g_{4E} \\
&= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_2) & -\sin(\theta_2) & 0 \\ 0 & \sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_3) & -\sin(\theta_3) & l_1 \\ 0 & \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\quad \cdots \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_4) & -\sin(\theta_4) & l_2 \\ 0 & \sin(\theta_4) & \cos(\theta_4) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

What if we do product in shorthand?

$$
\begin{aligned}
g_E &= \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{12} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{23} & p_{23} \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} R_{34} & p_{34} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{4E} & p_{4E} \\ 0 & 1 \end{bmatrix}} \\
&= \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{12} & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} R_{23} & p_{23} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{34}R_{4E} & R_{34}p_{4E} + p_{34} \\ 0 & 1 \end{bmatrix}} \\
&= \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} R_{12} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{23}R_{34}R_{4E} & R_{23}R_{34}p_{4E} + R_{23}p_{34} + p_{23} \\ 0 & 1 \end{bmatrix}} \\
&= \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{12}R_{23}R_{34}R_{4E} & R_{12}R_{23}R_{34}p_{4E} + R_{12}R_{23}p_{34} + R_{12}p_{23} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R_{01}R_{12}R_{23}R_{34}R_{4E} & R_{01}R_{12}R_{23}R_{34}p_{4E} + R_{01}R_{12}R_{23}p_{34} + R_{01}R_{12}p_{23} + p_{01} \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Note: try to do this calculation yourself in Mathematica or Matlab. You should arive at:

$$
g_E = \left[\begin{array}{ccc|c}
\cos(\theta_1) & -\sin(\theta_1)\cos(\theta_2+\theta_3+\theta_4) & \sin(\theta_1)\sin(\theta_2+\theta_3+\theta_4) & -\sin(\theta_1)\,(l_1\cos(\theta_2)+l_2\cos(\theta_2+\theta_3)+l_3\cos(\theta_2+\theta_3+\theta_4)) \\
\sin(\theta_1) & \cos(\theta_2+\theta_3+\theta_4) & -\cos(\theta_2)\sin(\theta_2+\theta_3+\theta_4) & \cos(\theta_1)(l_1\cos(\theta_2)+l_2\cos(\theta_2+\theta_3)+l_3\cos(\theta_2+\theta_3+\theta_4)) \\
0 & \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_2+\theta_3+\theta_4) & l_0+l_1\sin(\theta_2)+l_2\sin(\theta_2+\theta_3)+l_3\sin(\theta_2+\theta_3+\theta_4) \\
\hline
0 & 0 & 0 & 1
\end{array}\right]
$$