Topics Covered:

- Definitions
- Transformations

Additional Reading:

- Lynch, K.M. and Park, F.C. Modern Robotics: Section 3.0-3.1
- Craig, J.J. Introduction to Robotics: Section 2.1-2.3
- Murray et al. A Mathematical Introduction to Robotic Manipulation: Section 2.1

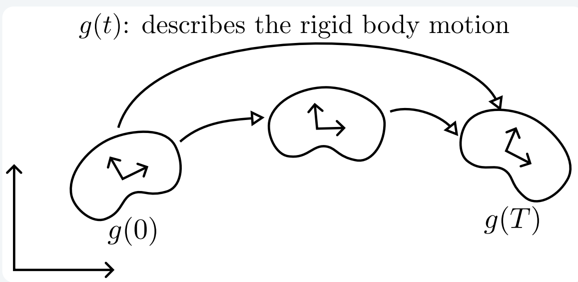
Definitions

Definition: Rigid Body

A rigid body is a collection of points/particles which have a fixed relationship amongst themselves.

Definition: Rigid Body Motion

A rigid body motion describes how the individual particles of a rigid body move as a function of time. Equivalently, a rigid body motion is the motion of the body fixed reference frame.



Definition: Displacement/Transformation

A displacement or transformation is the movement or motion of a rigid body without reference to time-scale.

- from g(0) to g(T) is displacement

Definition: Planar Rigid Body Motion

A planar rigid body motion is rigid body motion during which all of the particles remain in a plane (or a set of parallel planes).

Definition: Degrees of Freedom

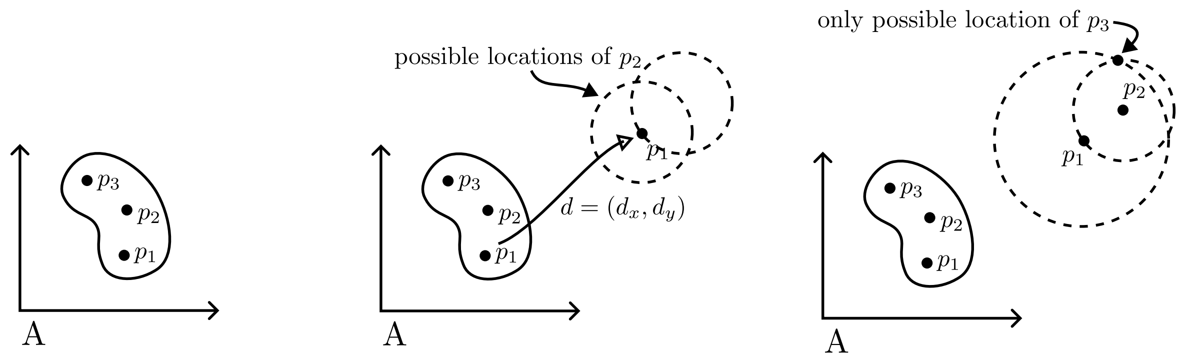
The degrees of freedom (DOF) of a motion is the minimal number of independent variables needed to uniquely specify the motion of an object.

- Why important? Because the variables we'll use sometimes have a bigger dimension than the DOF.

Question: How many degrees of freedom in planar motion?

Proposition 1. *Planar motion has 3 degrees of freedom.*

Proof. Consider the planar rigid body illustrated in Figure 1.



(a) rigid body (b) constraints on p_1 and p_2 (c) constraints on p_3

Figure 1: Planar Rigid Body considered in Proposition 1.

1. The location of particle p_i denoted by (x_i, y_i). Lets consider how each reference point would be defined by a displacement of the rigid body as shown in the next figure.
2. Consider p_1. It has 2 degrees of freedom since $d = (d_x, d_y)$ is arbitrary.
3. consider p_2. It must satisfy some fixed relationship with respect to p_1, by definition of rigid body.
   $\implies$ i.e., fixed distance $(x_2 - x_1)^2 + (y_2 - y_1)^2 = d_{12}^2$.
   We started with 2 DOF for p_2 but we added one constraint
   $\implies$ 2 DOF - 1 constraint = 1 DOF.
4. Consider p_3. It must satisfy some fixed relationship with respect to p_1 and p_2.
   $\implies$ i.e., fixed distance $(x_3 - x_1)^2 + (y_3 - y_1)^2 = d_{13}^2$ and $(x_3 - x_2)^2 + (y_3 - y_2)^2 = d_{23}^2$.
   2 DOF - 2 constraints = 0 DOF.

5. consider $p_i, i > 3$.
   2 DOF, $i-1$ constraints.
   $\implies$ 2 real constraints, $i-3$ redundant ones
   $\implies$ 2 DOF - 2 constraints = 0 DOF.

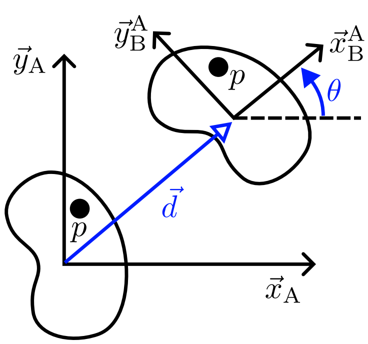
$\implies$ 3 total degrees of freedom. In general, will have $2n - (2(n-2)+1)$ constraints = 3 DOF. (for n particles).

□

Meaning: If I want to describe the configuration associated to a planar rigid body, what do I need? $\implies$ (x, y, θ).

Coordinates

The 3DOF required to represent a planar rigid body are position and orientation. In a sense, coordinates also describe a displacement or transformation.



Configuration of a rigid body:

$$g = (x, y, \theta)^\top \qquad \text{(Vector Notation)}$$

or equivalently:

$$g = (x, y, R(\theta))$$
$$g = (\vec{d}, R(\theta))$$
$$g = (\vec{d}, \theta)$$

Note: $R(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Note that we will use the notation $\vec{a}$ and curly brackets to denote vectors $\left(\vec{a} = \begin{Bmatrix} x \\ y \end{Bmatrix}\right)$.